Chapter 297

Confidence Intervals for Cpk

Introduction

This routine calculates the sample size needed to obtain a specified width of a Cpk confidence interval at a stated confidence level.

Cpk is a process capability index used to measure what a process is capable of producing. Unlike Cp, Cpk makes no assumption that the process mean is centered between the specification limits. Cpk requires the assumption that the measurements are normally distributed.

The formula for the calculation of Cpk is

$$Cpk = \min(USL - \mu, \mu - LSL) / (3\sigma)$$

where USL and LSL are the upper and lower specification limits, respectively.

A process with a Cpk of 2.0 is considered excellent, while one with a Cpk of 1.33 is considered adequate.

Technical Details

This procedure is based on the results of Mathews (2010). A 100(1 – α)% confidence interval for Cpk is given by

$$P\left(\widehat{Cpk}\left[1 - z_{1-\alpha/2}\sqrt{\frac{1}{n}\left(\frac{1}{9\widehat{Cpk}^2} + \frac{1}{2}\right)}\right] \le Cp \le \widehat{Cpk}\left[1 + z_{1-\alpha/2}\sqrt{\frac{1}{n}\left(\frac{1}{9\widehat{Cpk}^2} + \frac{1}{2}\right)}\right]\right) = 1 - \alpha$$

where $\widehat{Cpk}$ is the estimated value of Cpk, n is the sample size, and $z_{1-\alpha/2}$ is the specific value of the standard normal random variate that has probability $1 - \alpha/2$ to the left.

One-sided limits may be obtained by replacing $\alpha/2$ by α.

Confidence Interval Width

The confidence interval width, confidence level, and sample size are related in the equation

$$Width = Cpk_{upper} - Cpk_{lower}$$

This equation can be used to find n, α, or the width.

Confidence Level

The confidence level, $1 - \alpha$, has the following interpretation. If thousands of samples of *n* items are drawn from a population using simple random sampling and a confidence interval is calculated for each sample, the proportion of those intervals that will include the true population parameter is $1 - \alpha$.

Example 1 – Calculating Sample Size

Suppose a study is planned in which the researcher wishes to construct a two-sided 95% confidence interval for Cpk such that the width of the interval is no wider than 0.10. The researcher would like to examine Cpk values of 1.0, 1.5, 2.0, and 3.0 to determine the effect of the Cpk estimate on necessary sample size.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For	**Sample Size**
Interval Type	**Two-Sided**
Confidence Level (1 – Alpha)	**0.95**
Confidence Interval Width (Two-Sided)	**0.10**
Cpk	**1 1.5 2 3**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Reports

Numeric Results

Solve For: Sample Size
Interval Type: Two-Sided

Confidence Level	Sample Size N	Confidence Interval Width		Cpk	Confidence Interval Limits	
		Target	Actual		Lower	Upper
0.95	940	0.1	0.1	1.0	0.95	1.05
0.95	1900	0.1	0.1	1.5	1.45	1.55
0.95	3244	0.1	0.1	2.0	1.95	2.05
0.95	7086	0.1	0.1	3.0	2.95	3.05

Confidence Level	The proportion of confidence intervals (constructed with this same confidence level, sample size, etc.) that would contain the true value of Cpk.
N	The size of the sample drawn from the population.
Confidence Interval Width	The distance between the lower and upper confidence interval limits.
Target Width	The width that was requested.
Actual Width	The calculated width. This is slightly different from the Target Width because N is an integer.
Cpk	Equal to min(USL - μ, μ - LSL) / 3σ, where USL and LSL are the upper and lower specification limits, μ is the process mean, and σ is the process standard deviation.
Confidence Interval Limits	The confidence interval of Cpk.

Summary Statements

A two-sided 95% confidence interval for Cpk is needed. The formulation of Mathews (2010) will be used to calculate the confidence interval. The sample Cpk is assumed to be 1. To produce a confidence interval with a width of no more than 0.1, a sample size of 940 will be needed.

Dropout-Inflated Sample Size

Dropout Rate	Sample Size N	Dropout-Inflated Enrollment Sample Size N'	Expected Number of Dropouts D
20%	940	1175	235
20%	1900	2375	475
20%	3244	4055	811
20%	7086	8858	1772

Dropout Rate	The percentage of subjects (or items) that are expected to be lost at random during the course of the study and for whom no response data will be collected (i.e., will be treated as "missing"). Abbreviated as DR.
N	The evaluable sample size at which the confidence interval is computed. If N subjects are evaluated out of the N' subjects that are enrolled in the study, the design will achieve the stated confidence interval.
N'	The total number of subjects that should be enrolled in the study in order to obtain N evaluable subjects, based on the assumed dropout rate. After solving for N, N' is calculated by inflating N using the formula N' = N / (1 - DR), with N' always rounded up. (See Julious, S.A. (2010) pages 52-53, or Chow, S.C., Shao, J., Wang, H., and Lokhnygina, Y. (2018) pages 32-33.)
D	The expected number of dropouts. D = N' - N.

Dropout Summary Statements

Anticipating a 20% dropout rate, 1175 subjects should be enrolled to obtain a final sample size of 940 subjects.

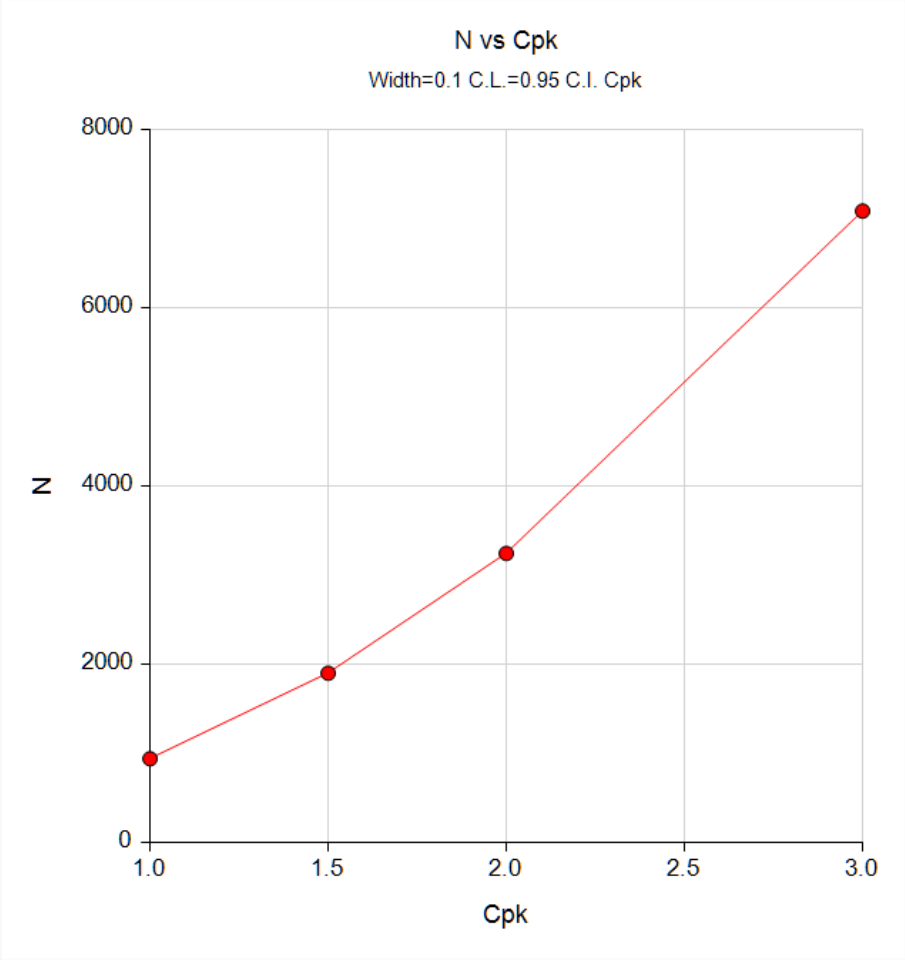
References

Kotz, S. and Johnson, N. 1993. Process Capability Indices. Chapman & Hall.
Mathews, Paul. 2010. Sample Size Calculations: Practical Methods for Engineers and Scientists. Mathews Malnar and Bailey, Inc.

This report shows the calculated sample size for each of the scenarios.

Plots Section

Plots



This plot shows the sample size versus Cpk.

Example 2 – Validation using Mathews (2010)

Mathews (2010), page 230, gives an example of a sample size calculation. In this example the value of Cpk is 1.0, the confidence level is 90%, and the width is 0.10. The resulting sample size is 662. Note that Mathews uses a normal approximation to the chi-square distribution which may make his results a little different than ours.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For	**Sample Size**
Interval Type	**Two-Sided**
Confidence Level (1 – Alpha)	**0.90**
Confidence Interval Width (Two-Sided)	**0.10**
Cpk	**1**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Solve For: Sample Size
Interval Type: Two-Sided

Confidence Level	Sample Size N	Confidence Interval Width		Cpk	Confidence Interval Limits	
		Target	Actual		Lower	Upper
0.9	662	0.1	0.1	1	0.95	1.05

PASS also calculates the sample size to be 662.