Chapter 407

Confidence Intervals for an Exponential Lifetime Percentile

Introduction

This routine calculates the number of events needed to obtain a specified width of a confidence interval for a percentile of an exponential distribution at a given level of confidence. The calculations assume Type-II censoring, that is, the experiment is run until a set number of events occur.

Technical Details

This procedure is based on the results of Mathews (2010) and Lawless (2003). The exponential lifetime model is based on the exponential density function

$$f(t) = \frac{1}{\theta}\exp(-t/\theta), \ \ t \geq 0$$

where θ is the mean lifetime, mean failure time, mean time to failure, or mean time between failures. This model is also parameterized in terms of failure rate, λ which is equal to $1/\theta$. In this case, the density is

$$f(t) = \lambda\exp(-\lambda t), \ \ t \geq 0.$$

The cumulative exponential distribution is

$$F(t) = 1 - \exp(-t/\theta), \ \ t \geq 0.$$

The survival or reliability function is

$$R(t) = 1 - F(t)$$

which in the case of the exponential distribution results in

$$R(t) = \exp(-t/\theta)$$

With the assumption of Type II censoring, the maximum-likelihood estimate of θ based on observing E failures in N items tested is

$$\bar{t} = \frac{1}{E}\sum_{k=1}^{N} t_k$$

where t_k is the amount of time that the kth item was under test, whether the event of interest was observed in that item or not.

An exact 100(1 – α) % confidence interval for the $100p^{th}$ percentile of lifetime, T_p, is given by

$$P\left(\frac{2Et_p}{\chi^2_{1-\alpha/2}} \le T_p \le \frac{2Et_p}{\chi^2_{\alpha/2}}\right) = 1 - \alpha$$

where χ^2_φ is the value of the chi-square random variate with *2E* degrees of freedom that has probability φ to the left. The value of t_p is calculate using

$$t_p = \bar{t}\ln(1-p)$$

One-sided limits may be obtained by replacing $\alpha/2$ by α.

Confidence Interval Width

The confidence interval width, confidence level, and number of events are related in the equation

$$Width = UCL - LCL$$

where LCL and UCL are the lower and upper confidence limits.

This equation can be used to find *E*, α, or the width.

Confidence Level

The confidence level, 1 – α, has the following interpretation. If thousands of samples of *n* items are drawn from a population using simple random sampling and a confidence interval is calculated for each sample, the proportion of those intervals that will include the true population parameter is 1 – α.

Example 1 – Calculating Sample Size

Suppose a study is planned in which the researcher wishes to construct a two-sided 95% confidence interval for *Tp* such that the width of the interval is 0.10 when 100p is 20%, 30%, or 40%. The percent censored is anticipated to be about 20% and the mean lifetime is 1.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For ....**Number of Events**
Interval Type ....**Two-Sided**
Confidence Level (1 - Alpha) **0.95**
Percent Censored.......................................**20**
Confidence Interval Width.............................**0.1**
θ (Mean Lifetime).......................................**1**
100p (Percentile Percentages) **20 30 40**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Reports

Numeric Results
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Solve For: Number of Events
Interval Type: Two-Sided
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Confidence Level	Number of Events E	Expected Sample Size N	Confidence Interval Width		Mean Lifetime θ	Percentile Percentage 100p	Lifetime Percentile tp	Confidence Interval Limits	
			Target	Actual				Lower	Upper
0.95	81	102	0.1	0.09945	1	20	0.22314	0.18153	0.28099
0.95	200	250	0.1	0.09979	1	30	0.35667	0.31198	0.41177
0.95	405	507	0.1	0.09996	1	40	0.51083	0.46451	0.56447

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Confidence Level	The proportion of confidence intervals (constructed with this same confidence level) that would contain the true value of tp.
E	The number of events that must occur before the experiment can be stopped.
N	Expected Sample Size. The anticipated number of subjects that must be sampled so that the desired number of events occur.
Confidence Interval Width	The distance between the lower and upper confidence interval limits.
Target Width	The width that was requested.
Actual Width	The calculated width. This is slightly different from the Target Width because E is an integer.
θ	Mean Lifetime. The total lifetime of all subjects divided by E, the number of events.
100p	The percentage of the percentile.
tp	Lifetime Percentile. The 100pth percentile of these exponentially distributed lifetimes.
Confidence Interval Limits	The confidence interval of tp.

Summary Statements
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A single-group design will be used to obtain a two-sided 95% confidence interval for the 20th percentile of an Exponential distribution. The Lawless (2003) and Mathews (2010) methods will be used in the confidence interval calculations. The percent censored is anticipated to be 20%. The mean lifetime is assumed to be 1. To produce a confidence interval with a width of no more than 0.1, 81 events will be needed. With 20% Type-II censoring, the sample size corresponding to 81 events is 102, and it is assumed that the experiment is run until 81 events occur.
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References
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Lawless, Jerald F. 2003. Statistical Models and Methods for Lifetime Data, 2nd Edition. John Wiley, New York.
Mathews, Paul. 2010. Sample Size Calculations: Practical Methods for Engineers and Scientists. Mathews Malnar
   and Bailey, Inc.
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This report shows the calculated sample size for each of the scenarios.

Plots Section

Plots
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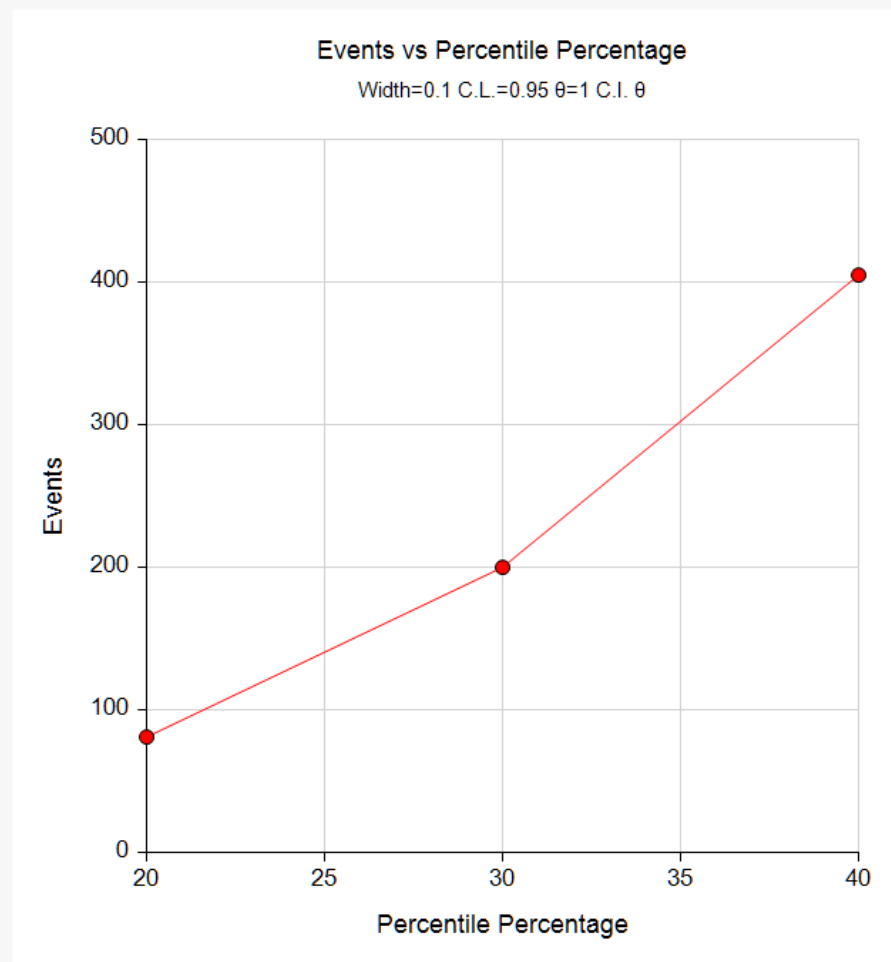
This plot shows the number of events for various percentile percentages.

Example 2 – Validation using Mathews (2010)

Mathews (2010), page 193, gives an example of a sample size calculation. In this example the value of θ is 1.0, the percentile percentage is 63.2 (the mean), the confidence level is 95%, and the width is 0.40 (plus or minus 0.2). The resulting number of events is 97. Note that Mathews uses a normal approximation to the chi-square distribution which makes his results a little different than ours.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab
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Solve For ....**Number of Events**
Interval Type..**Two-Sided**
Confidence Level (1 - Alpha) **0.95**
Percent Censored..**20**
Confidence Interval Width..............................**0.4**
θ (Mean Lifetime)..**1**
100p (Percentile Percentages) **63.2**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results
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Solve For: Number of Events
Interval Type: Two-Sided
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Confidence Level	Number of Events E	Expected Sample Size N	Confidence Interval Width		Mean Lifetime θ	Percentile Percentage 100p	Lifetime Percentile tp	Confidence Interval Limits	
			Target	Actual				Lower	Upper
0.95	100	125	0.4	0.39924	1	63.2	0.99967	0.8294	1.22864

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PASS calculates the number of events to be 100. The difference between 100 and the 97 that Mathews obtained is because Mathews used a normal approximation to the chi-square distribution. **PASS** used exact chi-square results.