Chapter 290

Control Charts for Means (Simulation)

Introduction

This procedure allows you to study the run length distribution of Shewhart (Xbar), Cusum, FIR Cusum, and EWMA process control charts for means using simulation. This procedure can also be used to study charts with a single observation at each sample. The in-control mean and standard deviation can be input directly or a specified number of in-control preliminary samples can be simulated based on a user-determined in-control distribution. The out-of-control distribution is flexible in terms of distribution type and distribution parameters. The Shewhart, Cusum, and EWMA parameters can also be flexibly input. This procedure can also be used to determine the necessary sample size to obtain a given run length.

Simulation Details

If the in-control mean and in-control standard deviation are assumed to be known, the steps to the simulation process are as follows (assume a sample consists of *n* observations).

1. An out-of-control sample of size *n* is generated according to the specified distribution parameters of the out-of-control distribution.
2. The average of the sample is produced and, if necessary for the particular type of control chart, the standard deviation.
3. Based on the control chart criteria, it is determined whether this sample results in an out-of-control signal.
4. If the sample results in an out-of-control signal, the sample number is recorded as the run length for that simulation. If the sample does not result in an out-of-control signal, return to Step 1.
5. Steps 1 through 4 are repeated until the number of simulations (*Nsim*) is reached. The result is *Nsim* run lengths.
6. The average or median or specified percentile of the run length distribution is reported.

If the in-control mean and in-control standard deviation are to be simulated based on in-control preliminary samples (*NPrelim*), the steps to the simulation process are as follows (assume a sample consists of *n* observations).

1. *NPrelim* in-control samples of size *n* are generated according to the specified distribution parameters of the in-control distribution.
2. The in-control average and standard deviation are calculated based on the *NPrelim* simulated in-control samples.
3. An out-of-control sample of size *n* is generated according to the specified distribution parameters of the out-of-control distribution.

4. The average of the sample is produced and, if necessary for the particular type of control chart, the standard deviation.
5. Based on the control chart criteria, it is determined whether this sample results in an out-of-control signal.
6. If the sample results in an out-of-control signal, the sample number is recorded as the run length for that simulation. If the sample does not result in an out-of-control signal, return to Step 3.
7. Steps 1 through 6 are repeated until the number of simulations (*Nsim*) is reached. The result is *Nsim* run lengths.
8. The average or median or specified percentile of the run length distribution is reported.

Data Distributions

A wide variety of distributions may be studied. These distributions can vary in skewness, elongation, or other features such as bimodality. A detailed discussion of the distributions that may be used in the simulation is provided in the chapter "Data Simulator".

Formulas for Constructing Control Charts

Suppose we have *k* subgroups, each of size *n*. Let x_{ij} represent the measurement in the j^{th} sample of the i^{th} subgroup. Three statistics that are routinely computed (depending on the type of control chart) for each subgroup are:

The subgroup mean

$$\bar{x}_i = \frac{\sum_{j=1}^{n} x_{ij}}{n}$$

the subgroup range

$$R_i = x_{(n)} - x_{(1)}$$

and/or the subgroup standard deviation

$$s_i = \sqrt{\frac{\sum_{j=1}^{n} (x_{ij} - \bar{x}_i)^2}{n-1}}$$

Estimating Sigma

Control limits vary according to the type of control chart used. These require an estimate of the process mean, μ_x (mu), and the process variability, σ_x (sigma). A known estimate of μ_x may be supplied by the user, or it may be estimated by the average of the averages of a number of in-control preliminary samples, 'x double bar' (also known as the grand mean):

$$\bar{\bar{x}}_i = \frac{\sum_{i=1}^{k} \bar{x}_{ij}}{k}$$

When sigma is not input directly, there are two methods available for estimating σ_x.

Method 1: Estimating Sigma from the Ranges

$$\hat{\sigma}_x = \frac{\bar{R}}{d_2}$$

where

$$\bar{R} = \frac{\sum_{i=1}^{k} R_i}{k}$$

$$d_2 = \frac{E(R)}{\sigma_x} = \frac{\mu_R}{\sigma_x}$$

Making the assumption that the x_{ij}'s follow the normal distribution with constant mean and variance, we can derive values for d_2 through the use of numerical integration.

Method 2: Estimating Sigma from the Standard Deviations

$$\hat{\sigma}_x = \frac{\bar{s}}{c_4}$$

where

$$\bar{s} = \frac{\sum_{i=1}^{k} s_i}{k}$$

$$c_4 = \frac{E(s)}{\sigma_x} = \frac{\mu_s}{\sigma_x}$$

Making the assumption that the x_{ij}'s follow the normal distribution with constant mean and variance, we can derive values for c_4 from the following formula.

$$c_4 = \sqrt{\frac{2}{n-1}}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

Estimating Sigma when n = 1

When n is one, we cannot calculate R_i or s_i since these require at least two measurements. In this case, we use the standard deviation of all k measurements. Unfortunately, this method does not approximate the within-subgroup variation. Rather, it combines the within and the between subgroup variation.

Xbar Chart Limits

The lower and upper control limits for the Xbar chart are calculated using the formula

$$LCL = \bar{\bar{x}} - z\left(\frac{\hat{\sigma}_x}{\sqrt{n}}\right)$$

$$UCL = \bar{\bar{x}} + z\left(\frac{\hat{\sigma}_x}{\sqrt{n}}\right)$$

where z is a multiplier (often set to three) chosen to reduce the possibility of false alarms (signaling an out-of-control situation when the process is in control).

Cusum and FIR Cusum Charts

The Cusum chart has been shown to detect small shifts in the process average much quicker than the Xbar chart.

In **PASS** we use the Cusum procedure presented by Ryan (1989). This procedure may be summarized as follows:

1. Calculate all statistics as if you were going to generate an Xbar chart.
2. Calculate the z_i using the formula

$$z_i = \frac{\bar{x}_i - \bar{\bar{x}}}{\hat{\sigma}_{\bar{x}}}$$

3. Calculate the lower and upper cumulative sums as follows

$$S_{Li} = -\max[0, (-z_i - K) + S_{Li-1}]$$

$$S_{Hi} = \max[0, (z_i - K) + S_{Hi-1}]$$

4. The control limits are chosen as plus or minus h. Often, K is set to 0.5 (for detecting one-sigma shifts in the mean) and h is set to 5.
5. Usually, the starting value for S_{Li} and S_{Hi} is zero. Occasionally, however, a “fast initial response” (FIR) value of $h/2$ is used.

EWMA Chart Limits

The lower and upper control limits for the exponentially weighted moving-average (EWMA) chart are calculated using the formula

$$LCL_i = \bar{\bar{x}} - L\left(\frac{\hat{\sigma}_x}{\sqrt{n}}\right)\sqrt{\frac{R}{2-R}\left[1-(1-R)^{2i}\right]}$$

$$UCL_i = \bar{\bar{x}} + L\left(\frac{\hat{\sigma}_x}{\sqrt{n}}\right)\sqrt{\frac{R}{2-R}\left[1-(1-R)^{2i}\right]}$$

where L is a multiplier (usually set to three) and R is smoothing constant.

Example 1 – Run Length Distribution

A researcher wishes to examine the run length distribution for a process monitored by a Shewhart (Xbar) chart. Six observations are to make up the sample examined at each hour. The in-control mean and standard deviation are known to be 5.2 and 3.1, respectively. The researcher would like to see the run length distribution if the out-of-control mean and standard deviation are 6.2 and 3.1, respectively. A Z-Multiplier of 3.0 is to be used in the control chart for the boundaries.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For .... **Run Length Distribution**
Simulations **5000**
Random Seed.. **4156386** (for Reproducibility)
Maximum Run Length.................................. **5000**
Run Length Summary 1 .... **ARL (Average Run Length)**
Run Length Summary 2 .... **MRL (Median Run Length)**
n (Sample Size) **6**

Distributions Tab

In-Control Distributions Specified By .... **Mean, Standard Deviation Directly**
In-Control Mean (Center Line) **5.2**
In-Control Standard Deviation....................... **3.1**
Out-of-Control Distribution .... **N(M1 S)**
M1 (Out-of-Control Mean)............................. **6.2**
Parameter 1 Label .... **S**
Parameter 1 Value(s).................................... **3.1**

Tests Tab

Shewhart (Xbar).. **Checked**
All Other Tests .... **Unchecked**
Use Z-Multiplier or Probability....................... **Z-Multiplier**
Z-Multiplier.. **3.0**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results for Control Charts for a Process Mean

Solve For: Run Length Distribution
Shewhart Control Limits: Determined by specifying a centerline mean and standard deviation directly
Out-of-Control Distribution (determines size of shift): Normal(M1 S)

							Control Limits			
Chart Type	Average Run Length	Median Run Length	Sample Size n	In-Control Mean	Z-Multiplier	In-Control SD	Lower LCL	Upper UCL	Out-of-Control Mean M1	S
Shewhart	72.9	51	6	5.2	3	3.1	1.4	9	6.2	3.1

Simulations: 5000. Run Time: 2.54 seconds.
User-Entered Random Seed: 4156386

Average Run Length	The mean of the run lengths across all simulations.
Median Run Length	The median of the run lengths across all simulations.
n	The number of units measured in each sample.
In-Control Mean	The assumed known value of the center line of the control chart.
Z-multiplier	The Z-multiplier which corresponds to the two-sided probability of a single sample mean outside the control limits.
In-Control SD	The assumed known standard deviation that is used in the calculation of limits.
LCL and UCL	The lower and upper control chart limits, respectively.
Out-of-Control Mean M1	The mean of the distribution from which out-of-control samples are drawn. It's difference from the Mean Line is the shift to detect.
Other Parameters	These parameters, often S (standard deviation), define the distribution from which out-of-control samples are drawn.

Summary Statements

An Xbar control chart will be used to monitor the process mean for an out-of-control signal. The chart is assumed to have a mean line at 5.2 and lower and upper Shewhart control limits of 1.4 and 9, respectively. With samples of size 6 from the (out-of-control) distribution Normal(M1 S) with mean 6.2, the average run length is 72.9 and the median run length is 51. These results are based on 5000 simulations (Monte Carlo samples).

Individual Summaries

Out-of-Control Distribution (determines size of shift):	Normal(M1 S)
Simulations:	5000
Random Seed:	4156386 (User-Entered)

Shewhart Details

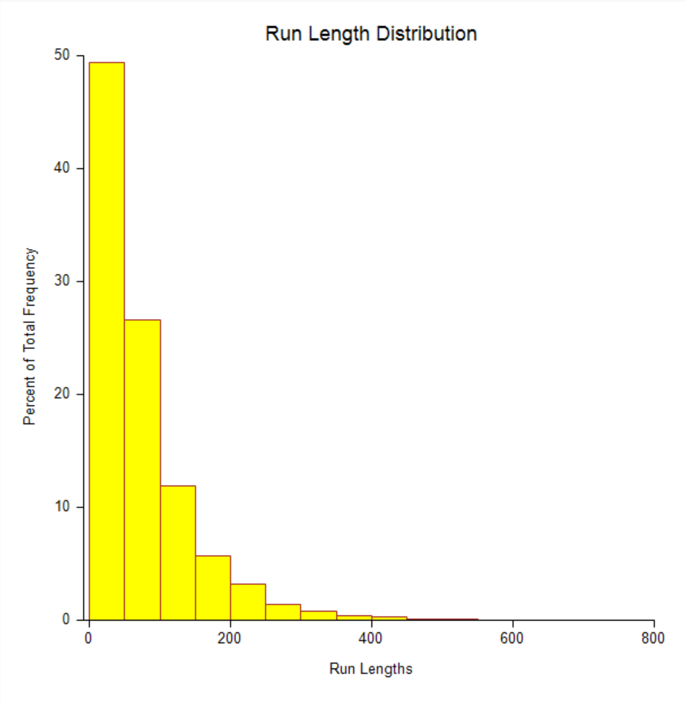
							Control Limits			
Chart Type	Average Run Length	Median Run Length	Sample Size n	In-Control Mean	Z-Multiplier	In-Control SD	Lower LCL	Upper UCL	Out-of-Control Mean M1	S
Shewhart	72.9	51	6	5.2	3	3.1	1.4	9	6.2	3.1

Average Run Length 95% CI: (70.8, 74.9)
Median Run Length 95% CI: (50, 53)

Average Run Length and Percentiles

Avg	1%	5%	10%	25%	50%	75%	90%	95%	99%
72.9	1	4	8	21	51	97	164	220	354

Histogram



References

Ryan, T.P. 1989. Statistical Methods for Quality Improvement. Wiley. New York.
Montgomery, D.C. 1991. Introduction to Statistical Quality Control. Wiley. New York.

The results will vary slightly because they are based on simulation. The plot shows the distribution of run lengths of 5000 simulated runs.

Example 2 – Comparing Tests

Continuing with the Example 1 parameters, the researchers would like to compare the various control chart tests available.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For ....**Run Length Distribution**
Simulations **5000**
Random Seed..**4205009** (for Reproducibility)
Maximum Run Length..................................**5000**
Run Length Summary 1................................**ARL (Average Run Length)**
Run Length Summary 2................................**MRL (Median Run Length)**
n (Sample Size) **6**

Distributions Tab

In-Control Distributions Specified By ....**Mean, Standard Deviation Directly**
In-Control Mean (Center Line) **5.2**
In-Control Standard Deviation.......................**3.1**
Out-of-Control Distribution ....**N(M1 S)**
M1 (Out-of-Control Mean).............................**6.2**
Parameter 1 Label ....**S**
Parameter 1 Value(s)....................................**3.1**

Tests Tab

All Tests...**Checked**
Use Z-Multiplier or Probability.......................**Z-Multiplier**
Z-Multiplier...**3.0**
K **0.5**
H **5**
FIR **2.5**
R **0.25**
L...**3**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results for Control Charts for a Process Mean

Solve For:	Run Length Distribution
Shewhart Control Limits:	Determined by specifying a centerline mean and standard deviation directly
Out-of-Control Distribution (determines size of shift):	Normal(M1 S)
Cusum Parameters:	K: 0.5, H: 5, FIR: 2.5
EWMA Parameters:	R: 0.25, L: 3

							Control Limits			
Chart Type	Average Run Length	Median Run Length	Sample Size n	In-Control Mean	Z-Multiplier	In-Control SD	Lower LCL	Upper UCL	Out-of-Control Mean M1	S
Shewhart	75.4	52	6	5.2	3	3.1	1.4	9	6.2	3.1
Cusum	15.4	13	6	5.2		3.1			6.2	3.1
Cus+Shew	14.7	12	6	5.2	3	3.1	1.4	9	6.2	3.1
FIR Cusum	10.1	7	6	5.2		3.1			6.2	3.1
FIR+Shew	9.8	7	6	5.2	3	3.1	1.4	9	6.2	3.1
EWMA	17.2	13	6	5.2		3.1			6.2	3.1
EWMA+Shew	16.5	13	6	5.2	3	3.1	1.4	9	6.2	3.1

Simulations: 5000. Run Time: 6.15 seconds.
User-Entered Random Seed: 4205009

The FIR tests show the process is out-of-control much sooner than the other tests.

Example 3 – Validation Using Montgomery (1991)

A table in Montgomery (1991), page 298, gives the average run lengths for a mean shift of one standard deviation to be Cusum (10.4), Cusum with Shewhart (10.20), Cusum with FIR (6.35), Cusum with FIR and Shewhart (6.32).

For reproducibility, we'll use a random seed of 4922450.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 3** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For ....**Run Length Distribution**
Simulations **100000**
Random Seed..**4922450** (for Reproducibility)
Maximum Run Length..................................**5000**
Run Length Summary 1................................**ARL (Average Run Length)**
Run Length Summary 2................................**MRL (Median Run Length)**
n (Sample Size) **1**

Distributions Tab

n-Control Distributions Specified By ....**Mean, Standard Deviation Directly**
In-Control Mean (Center Line) **0**
In-Control Standard Deviation.......................**1**
Out-of-Control Distribution ....**N(M1 S)**
M1 (Out-of-Control Mean).............................**1**
Parameter 1 Label ....**S**
Parameter 1 Value(s)....................................**1**

Tests Tab

FIR Cusum..**Checked**
Cusum ....**Checked**
FIR Cusum + Shewhart ....**Checked**
Cusum + Shewhart ....**Checked**
All Other Tests ....**Unchecked**
Use Z-Multiplier or Probability.......................**Z-Multiplier**
Z-Multiplier..**3.5**
K **0.5**
H **5**
FIR **2.5**
R **0.25**
L...**3**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results for Control Charts for a Process Mean

Solve For:	Run Length Distribution
Shewhart Control Limits:	Determined by specifying a centerline mean and standard deviation directly
Out-of-Control Distribution (determines size of shift):	Normal(M1 S)
Cusum Parameters:	K: 0.5, H: 5, FIR: 2.5

							Control Limits			
Chart Type	Average Run Length	Median Run Length	Sample Size n	In-Control Mean	Z-Multiplier	In-Control SD	Lower LCL	Upper UCL	Out-of-Control Mean M1	S
Cusum	10.35	9	1	0		1			1	1
Cus+Shew	10.26	9	1	0	3.5	1	-3.5	3.5	1	1
FIR Cusum	6.35	5	1	0		1			1	1
FIR+Shew	6.33	5	1	0	3.5	1	-3.5	3.5	1	1

Simulations: 100000. Run Time: 6.56 seconds.
User-Entered Random Seed: 4922450

The average run lengths are very close to those presented in Montgomery (1991).