

Chapter 804

Kendall's Tau-b Correlation Tests (Simulation)

Introduction

This procedure analyzes the power and significance level of the Kendall's Tau Correlation significance test using Monte Carlo simulation. This test is used to test whether the Kendall's Tau b correlation coefficient is non-zero. For each scenario that is set up, two simulations are run. One simulation estimates the significance level and the other estimates the power.

Kendall's Tau b is a popular statistic for describing the strength of the monotonic relationship between two variables. It ranges between plus and minus one.

Technical Details

Computer simulation allows us to estimate the power and significance level that is actually achieved by a test procedure in situations that are not mathematically tractable. Computer simulation was once limited to mainframe computers. But, in recent years, as computer speeds have increased, simulation studies can be completed on desktop and laptop computers in a reasonable period of time.

The steps to a simulation study are

1. Specify the test procedure and the test statistic. This includes the significance level, sample size, and underlying data distributions.
2. Generate a random sample of points (X, Y) from the bivariate distribution specified by the alternative hypothesis. Calculate the test statistic from the simulated data and determine if the null hypothesis is accepted or rejected. These samples are used to calculate the power of the test. In the case of paired data, the individual values are simulated are constructed so that they exhibit the specified amount of correlation.
3. Generate a second random sample of points (X, Y) from the bivariate distribution specified by the null hypothesis. Calculate the test statistic from the simulated data and determine if the null hypothesis is accepted or rejected. These samples are used to calculate the significance-level of the test. In the case of paired data, the individual values are simulated so that they exhibit the specified amount of correlation.
4. Repeat steps 2 and 3 several thousand times, tabulating the number of times the simulated data leads to a rejection of the null hypothesis. The power is the proportion of simulated samples in step 2 that lead to rejection. The significance level is the proportion of simulated samples in step 3 that lead to rejection.

Simulating Paired Distributions

In this routine, paired data may be generated from the bivariate normal distribution or from two specified marginal distributions. In the latter case, the simulation should mimic the actual data generation process as closely as possible, including the marginal distributions and the correlation between the two variables.

Obtaining paired samples from arbitrary distributions with a set correlation is difficult because the joint, bivariate distribution must be specified and simulated. Rather than specify the bivariate distribution, **PASS** requires the specification of the two marginal distributions and the correlation between them.

Monte Carlo samples with given marginal distributions and correlation are generated using the method suggested by Gentle (1998). The method begins by generating a large population of random numbers from the two distributions. Each of these populations is evaluated to determine if their means are within a small relative tolerance (0.0001) of the target mean. If the actual mean is not within the tolerance of the target mean, individual members of the population are replaced with new random numbers if the new random number moves the mean towards its target. Only a few hundred such swaps are required to bring the actual mean to within tolerance of the target mean.

The next step is to obtain the target correlation. This is accomplished by permuting one of the populations until they have the desired correlation.

The above steps provide a large pool (ten thousand to one million items) of random number pairs that exhibit the desired characteristics. This pool is then sampled at random using the uniform distribution to obtain the random number pairs used in the simulation.

This algorithm may be stated as follows.

1. Draw individual samples of size M from the two distributions where M is a large number over 10,000. Adjust these samples so that they have the specified mean and standard deviation. Label these samples A and B. Create an index of the values of A and B according to the order in which they are generated. Thus, the first value of A and the first value of B are indexed as one, the second values of A and B are indexed as two, and so on up to the final set which is indexed as M .
2. Compute the product-moment correlation between the two generated variates.
3. If the computed product-moment correlation is within a small tolerance (usually less than 0.001) of the specified correlation, go to step 7.
4. Select two indices (I and J) at random using uniform random numbers.
5. Determine what will happen to the product-moment correlation if B_I is swapped with B_J . If the swap will result in a correlation that is closer to the target value, swap the indices and proceed to step 6. Otherwise, go to step 4.
6. If the computed product-moment correlation is within the desired tolerance of the target correlation, go to step 7. Otherwise, go to step 4.
7. End with a population with the required marginal distributions and correlation.

A population created by this procedure tends to exhibit more variation in the tails of the distribution than in the center. Hence, the results using the bivariate normal option will be slightly different from those obtained when a custom model with two normal distributions is used.

Test Statistic

This section describes the test statistic that is used.

Kendall's Tau Correlation Coefficient

Kendall's Tau correlation coefficient is calculated from a sample of N data pairs (X, Y) by first creating a variable U as the ranks of X and a variable V as the ranks of Y (ties replaced with average ranks). Kendall's Tau is then calculated from U and V using

$$\hat{\tau} = \frac{2(n_C - n_D)}{\sqrt{N(N-1) - T_X} \sqrt{N(N-1) - T_Y}}$$

where

$$T_X = \sum_{i=1}^{S_X} (t_{(X)i}^2 - t_{(X)i})$$

$$T_Y = \sum_{i=1}^{S_Y} (t_{(Y)i}^2 - t_{(Y)i})$$

The parameter n_C is the total number of concordant pairs and n_D is the total number of discordant pairs. A pair of points (U_i, Y_i) and (U_j, Y_j) is said to be **concordant** when either of the following statements is true: 1. $(U_i < U_j)$ and $(Y_i < Y_j)$; 2. $(U_i > U_j)$ and $(Y_i > Y_j)$. Similarly, a pair of points is said to be **discordant** when either of the following statements is true: 1. $(U_i < U_j)$ and $(Y_i > Y_j)$; 2. $(U_i > U_j)$ and $(Y_i < Y_j)$. Pairs in which $(U_i = U_j)$ or $(Y_i = Y_j)$ are not classified as concordant or discordant and are ignored.

The value of $t_{(X)i}$ is the number of ties in the i^{th} set of ties of the X variables. There are S_X sets of ties in the X variable. The value of $t_{(Y)i}$ is the number of ties in the i^{th} set of ties of the Y variables. There are S_Y sets of ties in the Y variable.

Kendall's Tau Test Statistic

The statistical significance of Kendall's tau is tested using

$$z = \frac{S + \delta}{\sigma_S}$$

where

$$S = n_C - n_D$$

$$\sigma_S^2 = \frac{(N^2 - N)(2N + 5) - T_X'' - T_Y''}{18} + \frac{T_X' T_Y'}{9(N^2 - N)(N - 2)} + \frac{T_X T_Y}{2(N^2 - N)}$$

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$$T'_X = \sum_{i=1}^{S_X} (t_{(X)i}^2 - t_{(X)i}) (t_{(X)i} - 2)$$

$$T''_X = \sum_{i=1}^{S_X} (t_{(X)i}^2 - t_{(X)i}) (2t_{(X)i} + 5)$$

$$T'_Y = \sum_{i=1}^{S_X} (t_{(Y)i}^2 - t_{(Y)i}) (t_{(Y)i} - 2)$$

$$T''_Y = \sum_{i=1}^{S_X} (t_{(Y)i}^2 - t_{(Y)i}) (2t_{(Y)i} + 5)$$

$$\delta = \begin{cases} -1 & \text{if } S > 0 \\ 1 & \text{if } S < 0 \end{cases}$$

The distribution of z is approximately normal.

Test Procedure

The testing procedure is as follows. H_0 is the null hypothesis that τ is zero. H_1 represents the alternative hypothesis that the actual τ is non-zero. Choose a value z_α based on the normal distribution, so that the probability of rejecting H_0 when H_0 is true is equal to a specified value, α .

Select a sample of N items from the population and compute their test statistic z . If $z > z_\alpha$ reject the null hypothesis that $\tau = 0$ in favor of an alternative hypothesis that $\tau > 0$. The power is the probability of rejecting H_0 when $\tau = \tau_1$.

Unfortunately, the correlation that is entered into **PASS** is ρ_1 , not τ_1 .

Example 1 – Finding the Power

Suppose a study will be run to test whether Kendall's tau between forced vital capacity (X) and forced expiratory value (Y) in a particular population is non-zero. Find the power if $\rho_1 = 0.20$ and 0.30 , alpha is 0.05 , and $N = 20, 60, 100$.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

| | |
|---------------------------------|--------------------------------------|
| Solve For | Power |
| Alternative Hypothesis | $\tau \neq 0$ |
| Simulations | 5000 |
| Random Seed | 1023007 (for Reproducibility) |
| Alpha..... | 0.05 |
| N (Sample Size)..... | 20 60 100 |
| ρ_1 (Correlation H1) | 0.2 0.3 |
| Distribution..... | Bivariate Normal |

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Reports

Simulation Summary

| Variables | Simulation Distribution | |
|-------------------------|-------------------------|------------------|
| | Power (H1) | Alpha (H0) |
| X and Y | Bivariate Normal | Bivariate Normal |
| Random Number Pool Size | 10000 | |
| Number of Simulations | 5000 | |
| Random Seed | 1023007 (User-Entered) | |
| Run Time | 3.07 seconds | |

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Numeric Results

Solve For: **Power**Correlation Hypotheses: $H_0: \tau = 0$ vs. $H_1: \tau \neq 0$

| Scenario | Sample Size N | Power | Correlation | | Alpha | |
|----------|------------------|-------|-------------|----------|--------|--------|
| | | | ρ_0 | ρ_1 | Target | Actual |
| 1 | 20 | 0.116 | 0 | 0.2 | 0.05 | 0.050 |
| 2 | 60 | 0.286 | 0 | 0.2 | 0.05 | 0.051 |
| 3 | 100 | 0.445 | 0 | 0.2 | 0.05 | 0.055 |
| 4 | 20 | 0.226 | 0 | 0.3 | 0.05 | 0.046 |
| 5 | 60 | 0.612 | 0 | 0.3 | 0.05 | 0.053 |
| 6 | 100 | 0.857 | 0 | 0.3 | 0.05 | 0.052 |

| | |
|--------------|--|
| N | The size of the sample drawn from the population. It is the number of X-Y data points in a sample. |
| Power | The probability of rejecting a false null hypothesis. It is calculated by the power simulation. |
| ρ_0 | The Pearson correlation coefficient assuming the null hypothesis, H_0 , which is set to zero which results in a test of non-correlation between X and Y. |
| ρ_1 | The Pearson correlation coefficient assuming the alternative hypothesis, H_1 . This is the value at which the power is computed. |
| Target Alpha | The probability of rejecting a true null hypothesis. It is set by the user. |
| Actual Alpha | The alpha level that was actually achieved by the experiment. It is calculated by the alpha simulation. |

Summary Statements

A correlation (single group, Y versus X) design will be used to test whether the Kendall's Tau-b correlation is different from 0 ($H_0: \tau = 0$ versus $H_1: \tau \neq 0$). The comparison will be made using a two-sided, one-sample Kendall's Tau-b correlation Z-test, with a Type I error rate (α) of 0.05. Based on 5000 simulations of the bivariate normal distribution (under the alternative hypothesis), to detect a (Pearson) correlation of 0.2, with a sample size of 20, the power is 0.116.

Power and Alpha Confidence Intervals from Simulations

| Scenario | Sample Size N | Power | 95% Confidence Interval Limits for Power | | Alpha | | 95% Confidence Interval Limits for Alpha | |
|----------|------------------|-------|--|-------|--------|--------|--|-------|
| | | | Lower | Upper | Target | Actual | Lower | Upper |
| 1 | 20 | 0.116 | 0.107 | 0.125 | 0.05 | 0.050 | 0.044 | 0.056 |
| 2 | 60 | 0.286 | 0.274 | 0.299 | 0.05 | 0.051 | 0.045 | 0.057 |
| 3 | 100 | 0.445 | 0.432 | 0.459 | 0.05 | 0.055 | 0.048 | 0.061 |
| 4 | 20 | 0.226 | 0.214 | 0.237 | 0.05 | 0.046 | 0.041 | 0.052 |
| 5 | 60 | 0.612 | 0.598 | 0.626 | 0.05 | 0.053 | 0.047 | 0.059 |
| 6 | 100 | 0.857 | 0.848 | 0.867 | 0.05 | 0.052 | 0.046 | 0.058 |

| | |
|---|--|
| N | The size of the sample drawn from the population. It is the number of X-Y data points in a sample. |
| Power | The probability of rejecting H_0 when it is false. This is the actual value calculated by the power simulation. |
| Lower and Upper Limits of 95% C.I. of Power | The limits of an exact, 95% confidence interval for power based on the binomial distribution. They are calculated from the power simulation. |
| Target Alpha | The desired probability of rejecting a true null hypothesis at which the tests were run. |
| Actual Alpha | The alpha achieved by the test as calculated by the alpha simulation. |
| Lower and Upper Limits of 95% C.I. of Alpha | The limits of an exact, 95% confidence interval for alpha based on the binomial distribution. They are calculated from the alpha simulation. |

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Dropout-Inflated Sample Size

| Dropout Rate | Sample Size N | Dropout- Inflated Enrollment Sample Size N' | Expected Number of Dropouts D |
|--------------|------------------|---|--|
| 20% | 20 | 25 | 5 |
| 20% | 60 | 75 | 15 |
| 20% | 100 | 125 | 25 |

| | |
|--------------|--|
| Dropout Rate | The percentage of subjects (or items) that are expected to be lost at random during the course of the study and for whom no response data will be collected (i.e., will be treated as "missing"). Abbreviated as DR. |
| N | The evaluable sample size at which power is computed (as entered by the user). If N subjects are evaluated out of the N' subjects that are enrolled in the study, the design will achieve the stated power. |
| N' | The total number of subjects that should be enrolled in the study in order to obtain N evaluable subjects, based on the assumed dropout rate. N' is calculated by inflating N using the formula $N' = N / (1 - DR)$, with N' always rounded up. (See Julious, S.A. (2010) pages 52-53, or Chow, S.C., Shao, J., Wang, H., and Lokhnygina, Y. (2018) pages 32-33.) |
| D | The expected number of dropouts. $D = N' - N$. |

Dropout Summary Statements

Anticipating a 20% dropout rate, 25 subjects should be enrolled to obtain a final sample size of 20 subjects.

References

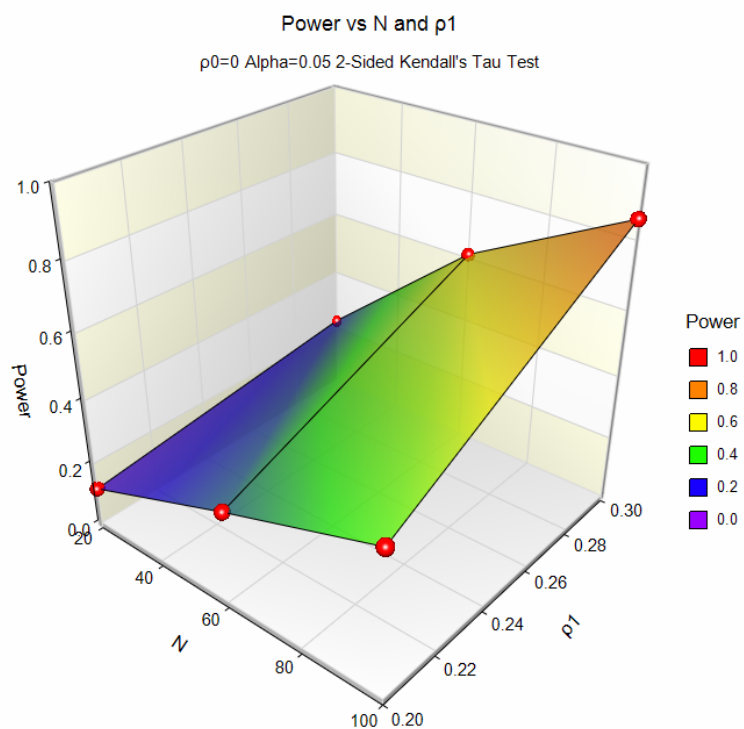
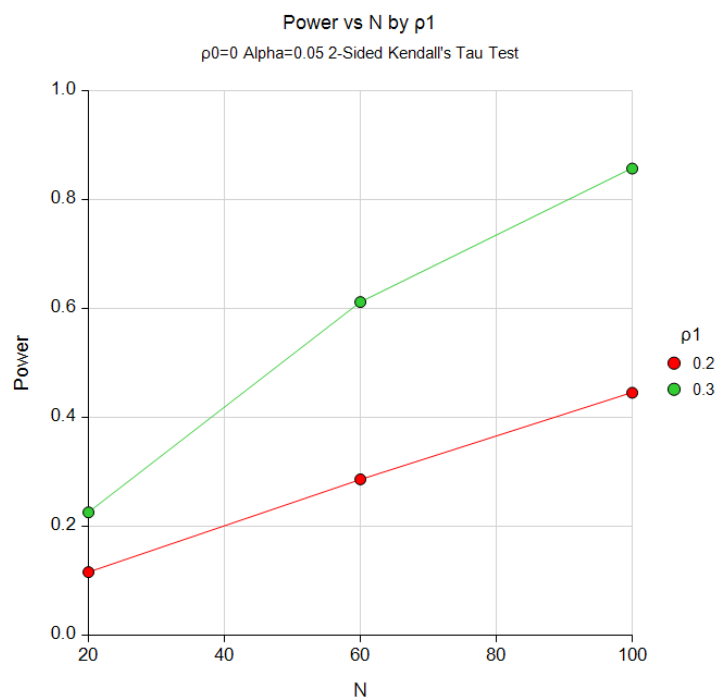
Kendall, M. and Gibbons, J.D. 1990. Rank Correlation Methods, 5th Edition. Oxford University Press. New York.
Devroye, Luc. 1986. Non-Uniform Random Variate Generation. Springer-Verlag. New York.

The Numeric Results report gives the basic results. The Power and Alpha Confidence Intervals report provides a confidence interval for each power and alpha value that was simulated. Definitions follow the reports.

Kendall's Tau-b Correlation Tests (Simulation)

Plots Section

Plots



These plots show the relationship between power and sample size in this example.

Example 2 – Validation using Zar (1984)

Zar (1984) page 312 presents an example in which the power of a Pearson's correlation coefficient is calculated. If $N = 12$, $\alpha = 0.05$, $\rho_0 = 0$, and $\rho_1 = 0.866$, Zar calculates a power of 98% for a two-sided test. We know that the power of Spearman's rho should be similar to that of Pearson's correlation.

For reproducibility, we'll use a random seed of 5838502.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

| | |
|---------------------------------|--------------------------------------|
| Solve For | Power |
| Alternative Hypothesis | $\tau \neq 0$ |
| Simulations | 50000 |
| Random Seed | 5838502 (for Reproducibility) |
| Alpha..... | 0.05 |
| N (Sample Size)..... | 12 |
| ρ_1 (Correlation H1) | 0.866 |
| Distribution | Bivariate Normal |

Output

Click the Calculate button to perform the calculations and generate the following output.

Simulation Summary

| Variables | Simulation Distribution | |
|-------------------------|-------------------------|------------------|
| | Power (H1) | Alpha (H0) |
| X and Y | Bivariate Normal | Bivariate Normal |
| Random Number Pool Size | 100000 | |
| Number of Simulations | 50000 | |
| Random Seed | 5838502 (User-Entered) | |
| Run Time | 1.26 seconds | |

Kendall's Tau-b Correlation Tests (Simulation)

Numeric ResultsSolve For: **Power**Correlation Hypotheses: $H_0: \tau = 0$ vs. $H_1: \tau \neq 0$

| Scenario | Sample Size N | Power | Correlation | | Alpha | |
|----------|---------------------|-------|-------------|----------|--------|--------|
| | | | ρ_0 | ρ_1 | Target | Actual |
| 1 | 12 | 0.943 | 0 | 0.866 | 0.05 | 0.045 |

The power of 0.943 is a little less than Zar's result, as we would expect.