

Chapter 594

One-Way Analysis of Variance Contrasts Allowing Unequal Variances

Introduction

This procedure computes power and sample size of non-zero null tests of contrasts of multiple means which are analyzed using the Welch-Satterthwaite t-test. This method is recommended when the group variances are not equal. The results in this chapter come from Jan and Shieh (2016).

Technical Details

Suppose G groups each have a normal distribution and with means $\mu_1, \mu_2, \dots, \mu_G$ and standard deviations $\sigma_1, \sigma_2, \dots, \sigma_G$. Let n_1, n_2, \dots, n_G denote the sample size of each group and let N denote the total sample size of all groups. The non-zero null F-test is used to show that the means are significantly different from each other. Sometimes, however, it is of interest to test a specific comparison or contrast of the means. This procedure provides results for contrasts used in an ANOVA design.

A *comparison* is a weighted average of the means, in which some of the weights may be negative. When the weights sum to zero, the comparison is called a *contrast*.

Suppose you want to test whether a linear contrast of the means is significantly different from zero. This contrast is defined as

$$\delta = \sum_{i=1}^G c_i \mu_i$$

Here c_1, c_2, \dots, c_G are the contrast coefficients. Note that $\sum_{i=1}^G c_i = 0$ is required. Also, to make different contrasts comparable, Kirk (2013) suggests that $\sum_{i=1}^G |c_i| = 2$.

An unbiased estimate of δ is found by replacing the population means by the corresponding sample means.

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The hypothesis testing of $H_0: \delta = \delta_0$ versus $H_1: \delta \neq \delta_0$ can be conducted using

$$T = \frac{\hat{\delta} - \delta_0}{\widehat{\sigma(\delta)}}$$

where

$$\delta_0 = \sum_{i=1}^G c_i \mu_{0i}$$

$$\widehat{\sigma^2(\delta)} = \sum_{i=1}^G c_i^2 S_i^2 / n_i$$

$$S_i^2 = \sum_{j=1}^{N_i} (X_{ij} - \bar{X}_i)^2 / (n_i - 1)$$

Here $\widehat{\sigma^2(\delta)}$ is the estimator of $\sigma^2(\delta) = \text{Var}(\hat{\delta})$, where

$$\sigma^2(\delta) = \sum_{i=1}^G c_i^2 \sigma_i^2 / n_i$$

Under the null hypothesis, Satterthwaite (1946) and Welch (1947) showed that T is approximately distributed as a Student's t with ν degrees of freedom, where

$$\nu = \frac{\left(\sum_{i=1}^G \frac{c_i^2 \sigma_i^2}{n_i} \right)^2}{\sum_{i=1}^G \frac{c_i^4 \sigma_i^4}{n_i^2 (n_i - 1)}}$$

This value is estimated as

$$\hat{\nu} = \frac{\left(\sum_{i=1}^G \frac{c_i^2 S_i^2}{n_i} \right)^2}{\sum_{i=1}^G \frac{c_i^4 S_i^4}{n_i^2 (n_i - 1)}}$$

Hence, the test statistic T has the approximate distribution

$$T \sim t_{\hat{\nu}}$$

The Welch-Satterthwaite test rejects H_0 at a significance level α if $|T| > t_{\hat{\nu}, 1 - \frac{\alpha}{2}}$.

Power

Shieh and Jan (2015) noted that T has the general approximate distribution

$$T \sim t_{\hat{\nu}, \Delta}$$

where $t_{\hat{\nu}, \Delta}$ is a noncentral t with $\hat{\nu}$ degrees of freedom and noncentrality parameter Δ . Here, Δ is defined as

$$\Delta = \frac{\delta_1 - \delta_0}{\sigma(\delta)}$$

where $\delta_1 = \sum_{i=1}^G c_i \mu_{1i}$.

Hence, the power can be approximated as

$$Power = P\left(|t_{\nu, \Delta}| > t_{\nu, 1-\frac{\alpha}{2}}\right)$$

When a sample size is desired, it can be determined using a standard binary search algorithm.

Example 1 – Finding Sample Size

This example will show all the reports that are available in the procedure. It will also show the impact on sample size of changing various options.

Suppose an experiment is being designed to assess the sample size needed for a one-way design with 3 groups that will be analyzed with various contrasts using the extended Welch test at a significance level of 0.05 and a power of 0.9. The null means are all 0. The alternative means are {1, 2, 4}. The standard deviations are {1, 3, 4}.

In order to showcase the use of the spreadsheet input, we will fill it with data so that it appears as follows.

Eq	SD	Con1	Con2	Con3
1	1	-1	0.5	0.5
1	3	0.5	-1	0.5
1	4	0.5	0.5	-1

Note that we have changed the default column names to more descriptive names. This is easily accomplished by right clicking on the name. We changed C1 to Eq, C2 to SD, C3 to Con1, C4 to Con2, and C5 to Con3.

The first two columns will be used as group allocations patterns: “Eq” is for all groups equal, and “SD” is for group sizes proportional to the standard deviations.

The last three columns hold various sets of contrast coefficients. The first compares group 1 to the average of groups 2 and 3. The second emphasizes group 2 and the third emphasizes group 3.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For **Sample Size**
 Alternative Hypothesis **Two-Sided ($H_1: \delta \neq \delta_0$)**
 Power..... **0.90**
 Alpha..... **0.05**
 G (Number of Groups) **3**
 Group Allocation Input Type **Enter Columns of r's (Allocation Patterns)**
 Columns of r's (Allocation Patterns)..... **EQ-SD**
 μ_0 Input Type..... **Enter μ_0 (Group Means|H0)**
 μ_0 (Group Means|H0) **0**
 μ_1 Input Type..... **Enter μ_1 (Group Means|H1)**
 μ_1 (Group Means|H1) **1 2 4**
 Contrast Input Type **Multiple Lists of Contrast Coefficients**
 Multiple Lists of Coefficients **CON1-CON3**
 σ Input Type..... **Enter σ (Group Standard Deviations)**
 σ (Group Standard Deviations) **1 3 4**

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Input Spreadsheet Data

Row	Eq	SD	Con1	Con2	Con3
1	1	1	-1.0	0.5	0.5
2	1	3	0.5	-1.0	0.5
3	1	4	0.5	0.5	-1.0

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Reports

Numeric Results

Solve For: [Sample Size](#)
 Number of Groups: 3
 Hypotheses: $H_0: \delta = \delta_0$ vs. $H_1: \delta \neq \delta_0$

Contrast Coefficients C	Power	Group Means											Alpha
		Sample Size			Means		Contrast		Group Standard Deviations σ	Standard Error of δ 's $\sigma(\delta)$	Noncentrality Parameter Δ		
					H0 μ_0	H1 μ_1	H0 δ_0	H1 δ_1					
		Total N	Allocation r	Group n									
Con1(1)	0.90158	60	Eq(1)	n(1)	$\mu_0(1)$	$\mu_1(1)$	0	2.0	$\sigma(1)$	0.602	3.322	0.05	
Con2(2)	0.90043	1674	Eq(1)	n(2)	$\mu_0(1)$	$\mu_1(1)$	0	0.5	$\sigma(1)$	0.154	3.245	0.05	
Con3(3)	0.90348	99	Eq(1)	n(3)	$\mu_0(1)$	$\mu_1(1)$	0	-2.5	$\sigma(1)$	0.749	-3.339	0.05	
Con1(1)	0.91365	64	SD(2)	n(4)	$\mu_0(1)$	$\mu_1(1)$	0	2.0	$\sigma(1)$	0.586	3.411	0.05	
Con2(2)	0.90046	1432	SD(2)	n(5)	$\mu_0(1)$	$\mu_1(1)$	0	0.5	$\sigma(1)$	0.154	3.245	0.05	
Con3(3)	0.90837	72	SD(2)	n(6)	$\mu_0(1)$	$\mu_1(1)$	0	-2.5	$\sigma(1)$	0.745	-3.354	0.05	

Item	Values
Con1(1)	-1, 0.5, 0.5
Con2(2)	0.5, -1, 0.5
Con3(3)	0.5, 0.5, -1
Eq(1)	1, 1, 1
SD(2)	1, 3, 4
n(1)	20, 20, 20
n(2)	558, 558, 558
n(3)	33, 33, 33
n(4)	8, 24, 32
n(5)	179, 537, 716
n(6)	9, 27, 36
$\mu_0(1)$	0, 0, 0
$\mu_1(1)$	1, 2, 4
$\sigma(1)$	1, 3, 4

- C The Contrast Coefficients is the name of the set containing the contrast coefficients. The only restriction is that the sum of the coefficients must be zero and $\delta_0 \neq \delta_1$.
- Power The probability of rejecting a false null hypothesis when the alternative hypothesis is true.
- N The total number of subjects in the study. N is the sum of the group sample sizes.
- r The name and number of the set containing the Group Allocation Pattern. These values are then rescaled so they sum to one to form the Group Allocation Proportions.
- n The Group Sample Size is the name and number of the set containing the sample size of each group.
- μ_0 The Group Means|H0 is the name and number of the set containing the group means under the null hypothesis. Note that $\delta_0 = \mu_0'C$.
- μ_1 The Group Means|H1 is the name and number of the set containing the group means under the alternative hypothesis. This is the set of means at which the power is calculated using $\delta_1 = \mu_1'C$.

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- δ_0 The Contrast Among Means|H0 or $\mu_0'C$ is the dot product of μ_0 and C assumed by H0. The dot product is the sum of the products of the corresponding entries of the two sets of numbers. Note that you must have $\delta_0 \neq \delta_1$.
 δ_1 The Contrast Among Means|H1 or $\mu_1'C$ is the dot product of μ_1 and C assumed by H1. The dot product is the sum of the products of the corresponding entries of the two sets of numbers. Note that you must have $\delta_0 \neq \delta_1$.
 σ The name and number of the set containing the standard deviation of each group.
 $\sigma(\delta)$ The standard error of the δ 's is used in the calculation of Δ . Note that $\sigma(\delta)^2 = \sum[(C_j \sigma_j)^2 / n_j]$.
 Δ The NCP (noncentrality parameter), is used with the noncentral t-distribution to calculate the power. Note that $\Delta = (\delta_1 - \delta_0) / \sigma(\delta)$.
Alpha The probability of rejecting a true null hypothesis.

Group Sample Size Details

n	N	Group Sample Sizes	Group Allocation Proportions
n(1)	60	20, 20, 20	0.333, 0.333, 0.333
n(2)	1674	558, 558, 558	0.333, 0.333, 0.333
n(3)	99	33, 33, 33	0.333, 0.333, 0.333
n(4)	64	8, 24, 32	0.125, 0.375, 0.5
n(5)	1432	179, 537, 716	0.125, 0.375, 0.5
n(6)	72	9, 27, 36	0.125, 0.375, 0.5

Summary Statements

A one-way design with 3 groups will be used to test whether the contrast of the means (δ) is different from 0 (H0: $\delta = 0$ versus H1: $\delta \neq 0$, where the null value 0 is the value of the contrast applied to the null means '0, 0, 0'). The comparison will be made using a two-sided Welch t-test using the contrast coefficients -1, 0.5, 0.5, with a Type I error rate (α) of 0.05. The within-group standard deviations for the 3 groups are assumed to be 1, 3, 4. The noncentral t-distribution noncentrality parameter is 3.322. To detect group means of 1, 2, 4 (or a contrast of means value of 2), with 90% power, group subject counts of 20, 20, 20 (totaling 60 subjects) will be needed.

Dropout-Inflated Sample Size

Dropout Rate	Sample Size N	Dropout- Inflated Enrollment Sample Size N'	Expected Number of Dropouts D
20%	60	75	15
20%	1674	2093	419
20%	99	124	25
20%	64	80	16
20%	1432	1790	358
20%	72	90	18

- Dropout Rate The percentage of subjects (or items) that are expected to be lost at random during the course of the study and for whom no response data will be collected (i.e., will be treated as "missing"). Abbreviated as DR.
N The evaluable sample size at which power is computed. If N subjects are evaluated out of the N' subjects that are enrolled in the study, the design will achieve the stated power.
N' The total number of subjects that should be enrolled in the study in order to obtain N evaluable subjects, based on the assumed dropout rate. After solving for N, N' is calculated by inflating N using the formula $N' = N / (1 - DR)$, with N' always rounded up. (See Julious, S.A. (2010) pages 52-53, or Chow, S.C., Shao, J., Wang, H., and Lohknygina, Y. (2018) pages 32-33.)
D The expected number of dropouts. $D = N' - N$.

Dropout Summary Statements

Anticipating a 20% dropout rate, 75 subjects should be enrolled to obtain a final sample size of 60 subjects.

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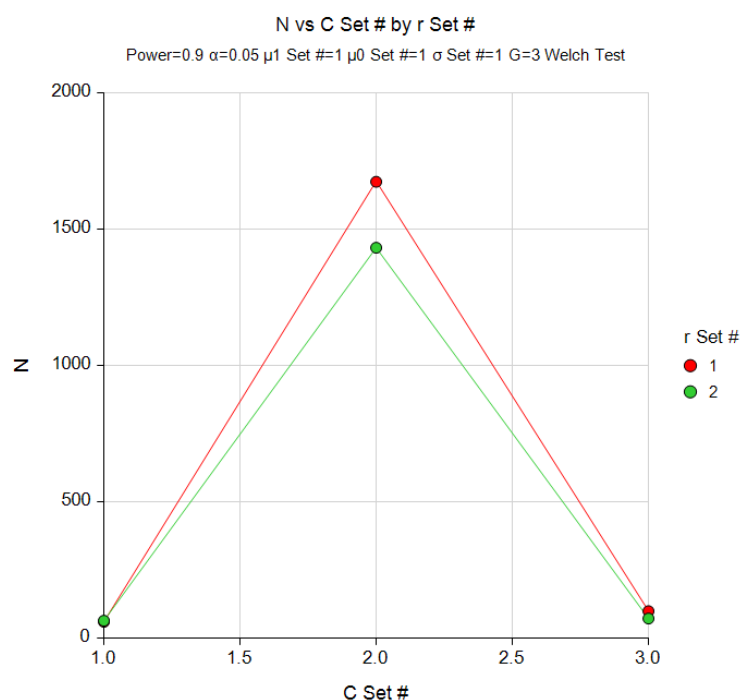
References

- Jan, S.L. and Shieh, G. 2016. 'A systematic approach to designing statistically powerful heteroscedastic 2 x 2 factorial studies while minimizing financial costs.' BMC Medical Research Methodology, 16:114.
- Kirk, Roger E. 2013. Experimental Design: Procedures for the Behavioral Sciences, 4th Edition. Sage. Washington, D.C.
- Luh, W.M. and Guo, J.H. 2016. 'Allocating sample sizes to reduce budget for fixed-effect 2 x 2 heterogeneous analysis of variance.' Journal of Experimental Education, 84:197-211.
- Satterthwaite, F.E. 1946. 'An approximate distribution of estimate of variance components,' Biometric Bulletin, 2:110-114.
- Shieh, G. and Jan, S-L. 2015. 'Power and sample size calculations for testing linear combinations of group means under variance heterogeneity with applications to meta and moderation analysis'. Psicologica, 36:367-390.
- Welch, B.L. 1951. 'On the Comparison of Several Mean Values: An Alternative Approach'. Biometrika, 38, 330-336.

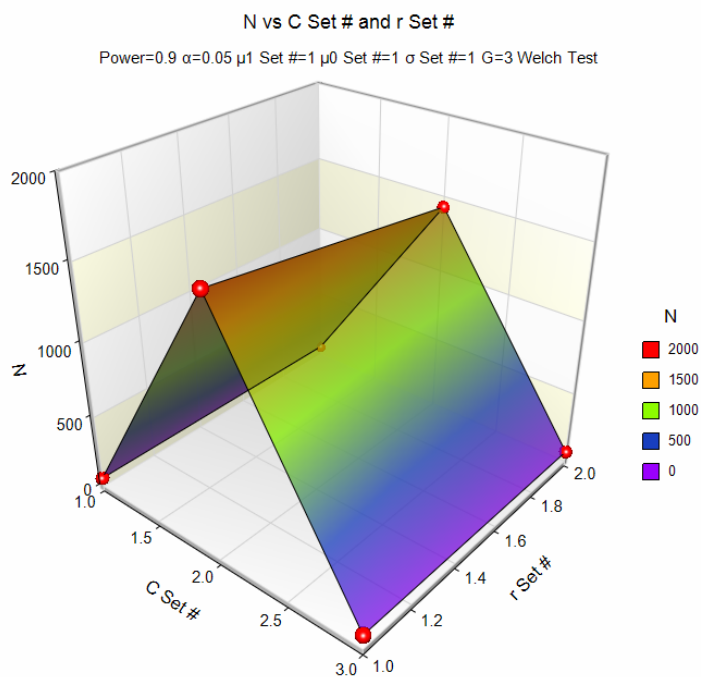
This report shows the numeric results of this study.

Plots Section

Plots



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The plots give a visual presentation of the results in the Numeric Report.

The main impression conveyed by this report and plot is that the required sample size is heavily impacted by the choice of the contrast coefficients, and, to a lesser degree, by the allocation pattern.

Example 2 – Validation using Jan and Shieh (2016)

Jan and Shieh (2016) page 6, Table 1, presents an example in which $\alpha = 0.05$, $G = 4$, the sample sizes are {16, 14, 7, 15}, the standard deviations are {0.83, 0.72, 0.34, 0.77}, the null means are {0, 0, 0, 0}, and the alternative means are {1.23, 0.42, 0.13, 0.38}. The contrast coefficients are {0.5, -0.5, -0.5, 0.5}. The resulting power is given as 0.8038.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For	Power
Alternative Hypothesis	Two-Sided ($H_1: \delta \neq \delta_0$)
Alpha.....	0.05
G (Number of Groups)	4
Group Allocation Input Type	Enter n (Group Sample Sizes)
n (Group Sample Sizes).....	16 14 7 15
μ_0 Input Type.....	Enter μ_0 (Group Means H0)
μ_0 (Group Means H0)	0
μ_1 Input Type.....	Enter μ_1 (Group Means H1)
μ_1 (Group Means H1)	1.23 0.42 0.13 0.38
Contrast Input Type	List of Contrast Coefficients
Contrast Coefficients.....	0.5 -0.5 -0.5 0.5
σ Input Type.....	Enter σ (Group Standard Deviations)
σ (Group Standard Deviations)	0.83 0.72 0.34 0.77

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Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Solve For: [Power](#)
 Number of Groups: 4
 Hypotheses: $H_0: \delta = \delta_0$ vs. $H_1: \delta \neq \delta_0$

Contrast Coefficients C	Power	Sample Size		Group Means		Contrast		Group Standard Deviations σ	Standard Error of δ 's $\sigma(\delta)$	Noncentrality Parameter Δ	Alpha
		Total	Group	H0	H1	H0	H1				
		N	n	μ_0	μ_1	δ_0	δ_1				
C(1)	0.80376	52	n(1)	$\mu_0(1)$	$\mu_1(1)$	0	0.53	$\sigma(1)$	0.184	2.873	0.05

Item Values

C(1) 0.5, -0.5, -0.5, 0.5
 n(1) 16, 14, 7, 15
 $\mu_0(1)$ 0, 0, 0, 0
 $\mu_1(1)$ 1.23, 0.42, 0.13, 0.38
 $\sigma(1)$ 0.83, 0.72, 0.34, 0.77

Group Sample Size Details

n	N	Group Sample Sizes	Group Allocation Proportions
n(1)	52	16, 14, 7, 15	0.308, 0.269, 0.135, 0.288

PASS also found the power to be 0.80376 which rounds to 0.8038. The procedure is validated.