Chapter 539

One-Way Analysis of Variance Contrasts Assuming Equal Variances

Introduction

The one-way (multiple group) design allows the means of two or more populations (groups) to be compared to determine if at least one mean is different from the others. The *F* test is used to determine statistical significance.

The usual F-test tests the hypothesis that all means are equal versus the alternative that at least one mean is different from the rest. Often, a more specific alternative is desired. For example, you might want to test whether the treatment means are different from the control mean, the low dose is different from the high dose, a linear trend exists across dose levels, and so on. These questions are tested using specific contrasts.

A *comparison* is a weighted average of the means, in which the weights may be negative. When the weights sum to zero, the comparison is called a *contrast*. **PASS** provides results for contrasts. To specify a contrast, we need only specify the weights.

For example, suppose an experiment conducted to study a drug will have three dose levels: none (control), 20 mg, and 40 mg. The first question is whether the drug made a difference. If it did, the average response for the two groups receiving the drug should be different from the control. If we label the group means M0, M2, and M4, we are interested in comparing M0 with M2 and M4. This can be done in two ways. One way is to construct two tests, one comparing M0 and M2 and the other comparing M0 and M4. Another method is to perform one test comparing M0 with the average of M2 and M4. These tests are conducted using contrasts. The coefficients are as follows:

M0 vs. M2

To compare M0 versus M2, use the coefficients -1, 1, 0. When applied to the group means, these coefficients result in the comparison M0(-1) + M2(1) + M4(0) which reduces to M2-M0. That is, this contrast results in the difference between two group means. We can test whether this difference is non-zero using the *t* test (or *F* test since the square of the *t* test is an *F* test).

M0 vs. M4

To compare M0 versus M4, use the coefficients -1, 0, 1. When applied to the group means, these coefficients result in the comparison M0(-1) + M2(0) + M4(1) which reduces to M4 - M0. That is, this contrast results in the difference between the two group means.

M0 vs. Average of M2 and M4

To compare M0 versus the average of M2 and M4, use the coefficients -2, 1, 1. When applied to the group means, these coefficients result in the comparison M0(-2) + M2(1) + M4(1) which is equivalent to M4 + M2 - 2(M0).

Assumptions

Using the *F* test requires certain assumptions. One reason for the popularity of the *F* test is its robustness in the face of assumption violation. However, if an assumption is not even approximately met, the significance levels and the power of the *F* test are invalidated. Unfortunately, in practice it often happens that several assumptions are not met. This makes matters even worse. Hence, steps should be taken to check the assumptions before important decisions are made.

The assumptions of the one-way analysis of variance are:

1. The data are continuous (not discrete).
2. The data follow the normal probability distribution. Each group is normally distributed about the group mean.
3. The variances within the groups are equal.
4. The groups are independent. There is no relationship among the individuals in one group as compared to another.
5. Each group is a simple random sample from its population. Each individual in the population has an equal probability of being selected in the sample.

Technical Details for One-Way ANOVA Contrasts

Suppose *G* groups each have a normal distribution and equal means ($\mu_1 = \mu_2 = \cdots = \mu_G$). Let $n_1 = n_2 = \cdots = n_G$ denote the number of subjects in each group and let *N* denote the total sample size of all groups. Let $\mu 1$ denote the weighted mean of all groups. That is

$$\mu 1 = \sum_{i=1}^{G} \left(\frac{n_i}{N}\right) \mu 1_i$$

Let σ denote the common standard deviation of all groups.

Suppose you want to test whether the contrast *C*

$$C = \sum_{i=1}^{G} c_i \mu 1_i$$

is significantly different from zero. Here the c_i's are the contrast coefficients.

Define

$$\sigma_C = \left|\sum_{i=1}^{G} c_i \mu 1_i\right| \Big/ \sqrt{N \sum_{i=1}^{G} \frac{c_i^2}{n_i}}$$

Define the noncentrality parameter λ_C, as

$$\lambda_C = N\sigma_C^2 / \sigma^2$$

Power Calculations for Contrasts

The calculation of the power of a test proceeds as follows:

1. Determine the critical value, $F_{1,N-G,\alpha}$, where α is the probability of a type-I error and G and N are defined above. Note that this is a two-tailed test as no direction is assigned in the alternative hypothesis.
2. From a hypothesized set of μ_i's, calculate the noncentrality parameter λ_C.
3. Compute the power as the probability of being greater than $F_{1,N-G,\alpha}$ on a noncentral-F distribution with noncentrality parameter λ_C.

Contrast Producing the Maximum Power

It is possible to calculate the coefficients of the contrast that will result in the maximum possible power. This contrast is based on a knowledge of the actual population means, so in practice it cannot be attained and may not be of practical use. However, this contrast lets you determine how close your power is compared with the maximum possible.

The contrast having the maximum power is mentioned in Winer (1991), page 151. The formula for the contrast coefficients is

$$c_j = n_j\left(\mu_j - \overline{\mu 1}\right) / \sqrt{\sum_{j=1}^{G} n_j\left(\mu 1_j - \overline{\mu 1}\right)^2}$$

Example 1 – Finding Power

An experiment is being designed to compare the means of four groups using a two-sided contrast test with a significance level of 0.05. The first group is a control group. The other three groups will have slightly different treatments applied. The researchers are mainly interested in whether the three treatment groups are different from the control group. Hence, they want to test the contrast represented by the coefficients {3, 1, 1, 1}. Treatment means {40, 10, 10, 10} represent clinically important group differences.

Previous studies have had standard deviations between 18 and 24. To better understand the relationship between power and sample size, the researcher wants to compute the power for several group sample sizes between 2 and 14. The sample sizes will be equal across all groups.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For	**Power**
Alternative Hypothesis	**Two-Sided (H1: δ ≠ 0)**
Alpha	**0.05**
G (Number of Groups)	**4**
Group Allocation Input Type	**Equal to ni (Sample Size per Group)**
ni (Sample Size Per Group)	**2 4 6 8 10 12 14**
μ1 Input Type	**Enter μ1 (Group Means|H1)**
μ1 (Group Means|H1)	**40 10 10 10**
Contrast Input Type	**List of Contrast Coefficients**
Contrast Coefficients	**-3 1 1 1**
σ (Standard Deviation)	**18 21 24**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Reports

Numeric Results

Solve For: Power
Number of Groups: 4
Hypotheses: H0: δ = 0 vs. H1: δ ≠ 0

Contrast Coefficients C	Power	Sample Size Total N	Sample Size Group ni	Group Means|H1 Means μ1	Group Means|H1 Contrast δ1	Standard Deviation σ	Standard Error of δ's σ(δ)	Noncentrality Parameter Δ	Alpha
C(1)	0.34714	8	2	μ1(1)	-90	18	2.449	-36.742	0.05
C(1)	0.27125	8	2	μ1(1)	-90	21	2.449	-36.742	0.05
C(1)	0.22011	8	2	μ1(1)	-90	24	2.449	-36.742	0.05
C(1)	0.75500	16	4	μ1(1)	-90	18	1.732	-51.962	0.05
C(1)	0.62314	16	4	μ1(1)	-90	21	1.732	-51.962	0.05
C(1)	0.51233	16	4	μ1(1)	-90	24	1.732	-51.962	0.05
C(1)	0.91944	24	6	μ1(1)	-90	18	1.414	-63.640	0.05
C(1)	0.82181	24	6	μ1(1)	-90	21	1.414	-63.640	0.05
C(1)	0.71320	24	6	μ1(1)	-90	24	1.414	-63.640	0.05
C(1)	0.97609	32	8	μ1(1)	-90	18	1.225	-73.485	0.05
C(1)	0.92175	32	8	μ1(1)	-90	21	1.225	-73.485	0.05
C(1)	0.84022	32	8	μ1(1)	-90	24	1.225	-73.485	0.05
C(1)	0.99342	40	10	μ1(1)	-90	18	1.095	-82.158	0.05
C(1)	0.96755	40	10	μ1(1)	-90	21	1.095	-82.158	0.05
C(1)	0.91476	40	10	μ1(1)	-90	24	1.095	-82.158	0.05
C(1)	0.99830	48	12	μ1(1)	-90	18	1.000	-90.000	0.05
C(1)	0.98714	48	12	μ1(1)	-90	21	1.000	-90.000	0.05
C(1)	0.95609	48	12	μ1(1)	-90	24	1.000	-90.000	0.05
C(1)	0.99958	56	14	μ1(1)	-90	18	0.926	-97.211	0.05
C(1)	0.99509	56	14	μ1(1)	-90	21	0.926	-97.211	0.05
C(1)	0.97803	56	14	μ1(1)	-90	24	0.926	-97.211	0.05

Item	Values
C(1)	-3, 1, 1, 1
μ1(1)	40, 10, 10, 10

C The name and number of the set containing the contrast coefficients. The only restrictions are that the sum of the coefficients must be zero and δ1 ≠ 0.
Power The probability of rejecting a false null hypothesis when the alternative hypothesis is true.
N The total number of subjects in the study. N is the sum of the group sample sizes.
ni The number of items sampled from each group.
μ1 The Group Means|H1 is the name and number of the set containing the group means under the alternative hypothesis. This is the set of means at which the power is calculated using δ1 = μ1'C ≠ 0, where C is the contrast vector.
δ1 The Contrast Among Means|H1 or μ1'C is the dot product of μ1 and C assumed by H1. The dot product is the sum of the products of the corresponding entries of the two sets of numbers. Note that you must have δ1 ≠ 0.
σ The common standard deviation of each group.
σ(δ) The standard error of the δ is used in the calculation of Δ. Note that $\sigma(\delta)^2 = \sigma^2\ \Sigma[(C(j))^2 / n(j)]$.
Δ The NCP (noncentrality parameter) is used with the noncentral t-distribution to calculate the power. Note that Δ = δ1 / σ(δ).
Alpha The probability of rejecting a true null hypothesis.

Group Sample Size Details

n	N	Group Sample Sizes	Group Allocation Proportions
n(1)	8	2, 2, 2, 2	0.25, 0.25, 0.25, 0.25
n(2)	16	4, 4, 4, 4	0.25, 0.25, 0.25, 0.25
n(3)	24	6, 6, 6, 6	0.25, 0.25, 0.25, 0.25
n(4)	32	8, 8, 8, 8	0.25, 0.25, 0.25, 0.25
n(5)	40	10, 10, 10, 10	0.25, 0.25, 0.25, 0.25
n(6)	48	12, 12, 12, 12	0.25, 0.25, 0.25, 0.25
n(7)	56	14, 14, 14, 14	0.25, 0.25, 0.25, 0.25

Summary Statements

A one-way design with 4 groups will be used to test whether the contrast of the means (δ) is different from 0 (H0: δ = 0 versus H1: δ ≠ 0). The comparison will be made using a two-sided t-test using the contrast coefficients -3, 1, 1, 1, with a Type I error rate (α) of 0.05. The group means under the null hypothesis are assumed to be equal. The (equal) within-group standard deviation for each of the 4 groups is assumed to be 18. The noncentral t-distribution noncentrality parameter is -36.742. To detect group means of 40, 10, 10, 10 (or a contrast of means value of -90), with group subject counts of 2, 2, 2, 2 (totaling 8 subjects), the power is 0.34714.

Dropout-Inflated Sample Size

Dropout Rate	Sample Size N	Dropout-Inflated Enrollment Sample Size N'	Expected Number of Dropouts D
20%	8	10	2
20%	16	20	4
20%	24	30	6
20%	32	40	8
20%	40	50	10
20%	48	60	12
20%	56	70	14

Dropout Rate The percentage of subjects (or items) that are expected to be lost at random during the course of the study and for whom no response data will be collected (i.e., will be treated as "missing"). Abbreviated as DR.
N The evaluable sample size at which power is computed (as entered by the user). If N subjects are evaluated out of the N' subjects that are enrolled in the study, the design will achieve the stated power.
N' The total number of subjects that should be enrolled in the study in order to obtain N evaluable subjects, based on the assumed dropout rate. N' is calculated by inflating N using the formula N' = N / (1 - DR), with N' always rounded up. (See Julious, S.A. (2010) pages 52-53, or Chow, S.C., Shao, J., Wang, H., and Lokhnygina, Y. (2018) pages 32-33.)
D The expected number of dropouts. D = N' - N.

Dropout Summary Statements

Anticipating a 20% dropout rate, 10 subjects should be enrolled to obtain a final sample size of 8 subjects.

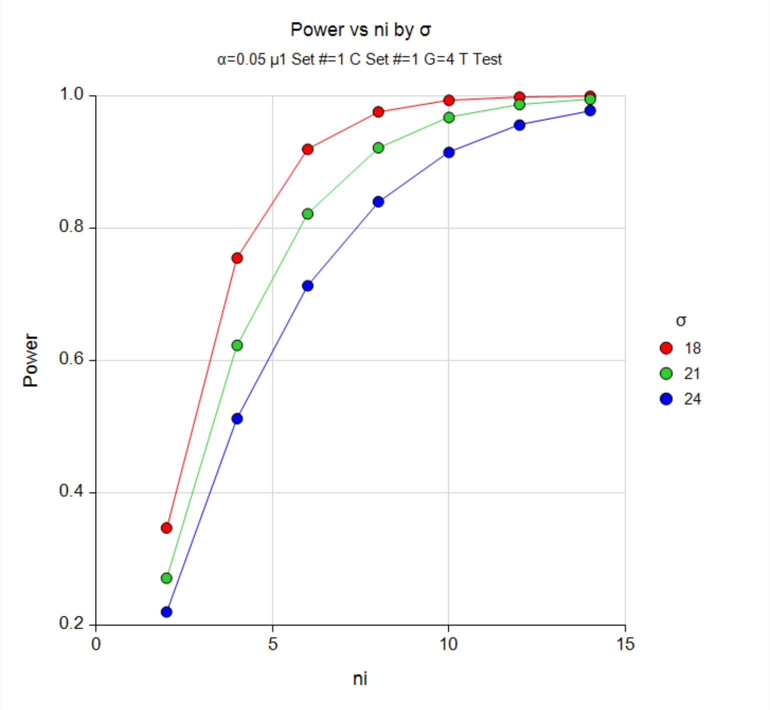
References

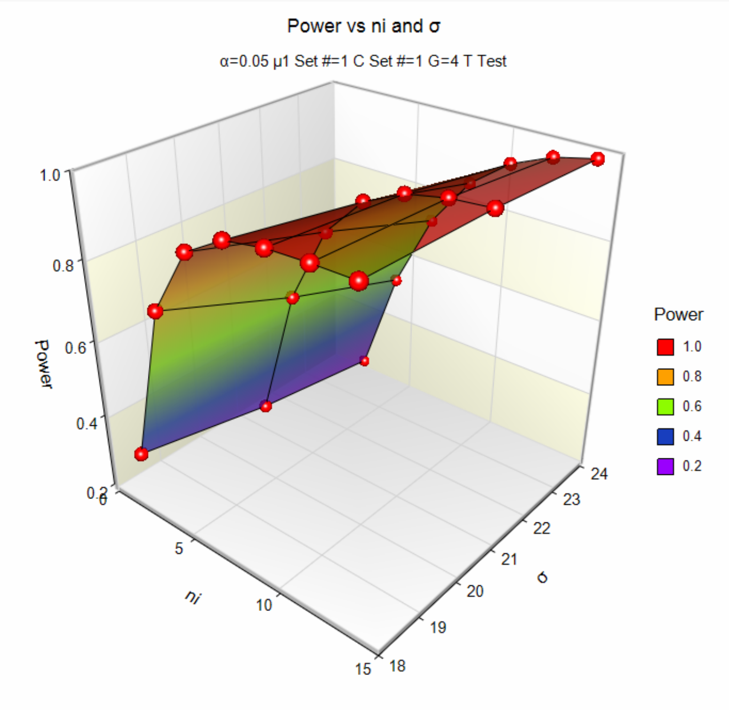
Desu, M. M. and Raghavarao, D. 1990. Sample Size Methodology. Academic Press. New York.
Fleiss, Joseph L. 1986. The Design and Analysis of Clinical Experiments. John Wiley & Sons. New York.
Kirk, Roger E. 1982. Experimental Design: Procedures for the Behavioral Sciences. Brooks/Cole. Pacific Grove, California.

This report shows the numeric results of this power study.

Plots Section

Plots





These plots give a visual presentation to the results in the Numeric Report. We can quickly see the impact on the power of increasing the sample size and the increase in the significance level.

When you create one of these plots, it is important to use trial and error to find an appropriate range for the horizontal variable so that you have results with both low and high power.

Example 2 – Validation using Hand Calculations

We will compute the following example by hand and then compare that with the results that **PASS** obtains. Here are the settings:

Alpha	**0.05**
G	**3**
Allocation	**Equal**
n	**5**
Means	**1, 2, 3**
K	**1**
Coefficients	**-2, 1, 1**
σ	**5**

Using these values, we find the following:

C'μ	**3**
σ_c^2	**9/18 = 0.5**
λ_C	**15 x 0.5/(25) = 0.3**
$F_{0.95,1,12}$	**4.747225**

Power = 0.0797

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For	**Power**
Alternative Hypothesis	**Two-Sided (H1: δ ≠ 0)**
Alpha	**0.05**
G (Number of Groups)	**3**
Group Allocation Input Type	**Equal to ni (Sample Size per Group)**
ni (Sample Size Per Group)	**5**
μ1 Input Type	**Enter μ1 (Group Means|H1)**
μ1 (Group Means|H1)	**1 2 3**
Contrast Input Type	**List of Contrast Coefficients**
Contrast Coefficients	**-2 1 1**
σ (Standard Deviation)	**5**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Solve For: Power
Number of Groups: 3
Hypotheses: H0: $\delta = 0$ vs. H1: $\delta \neq 0$

Contrast Coefficients C	Power	Sample Size Total N	Sample Size Group ni	Group Means|H1 Means μ1	Group Means|H1 Contrast δ1	Standard Deviation σ	Standard Error of δ's σ(δ)	Noncentrality Parameter Δ	Alpha
C(1)	0.07972	15	5	μ1(1)	3	5	1.095	2.739	0.05

Item	Values
C(1)	-2, 1, 1
μ1(1)	1, 2, 3

Group Sample Size Details

n	N	Group Sample Sizes	Group Allocation Proportions
n(1)	15	5, 5, 5	0.333, 0.333, 0.333

PASS has also calculated the power to be 0.0797.

Example 3 – Finding Sample Size for Various Allocation Patterns

Continuation of Example 1. An experiment is being designed to compare the means of four groups using a two-sided contrast test with a significance level of 0.05. The first group is a control group. The other three groups will have slightly different treatments applied. The researchers are mainly interested in whether the three treatment groups are different from the control group. Hence, they want to test the contrast represented by the coefficients {3, 1, 1, 1}. Treatment means {40, 10, 10, 10} represent clinically important group differences. Previous studies have had standard deviations between 18 and 24.

The researchers want to compare the sample size requirements for various sample allocation patterns: {1, 1, 1, 1}, {2, 1, 1, 1}, {3, 1, 1, 1}, {4, 1, 1, 1}. As you can see, these patterns allocate a progressively portion of the available participants to the control group. These patterns are entered into the spreadsheet as follows.

C1	C2	C3	C4
1	2	3	4
1	1	1	1
1	1	1	1
1	1	1	1

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 3** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For	**Sample Size**
Alternative Hypothesis	**Two-Sided (H1: δ ≠ 0)**
Power	**0.90**
Alpha	**0.05**
G (Number of Groups)	**4**
Group Allocation Input Type	**Enter Columns of r's (Allocation Patterns)**
Columns of r's (Allocation Patterns)	**1-4**
μ1 Input Type	**Enter μ1 (Group Means|H1)**
μ1 (Group Means|H1)	**40 10 10 10**
Contrast Input Type	**List of Contrast Coefficients**
Contrast Coefficients	**-3 1 1 1**
σ (Standard Deviation)	**18 21 24**

Input Spreadsheet Data

Row	C1	C2	C3	C4
1	1	2	3	4
2	1	1	1	1
3	1	1	1	1
4	1	1	1	1

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Solve For: Sample Size
Number of Groups: 4
Hypotheses: H0: δ = 0 vs. H1: δ ≠ 0

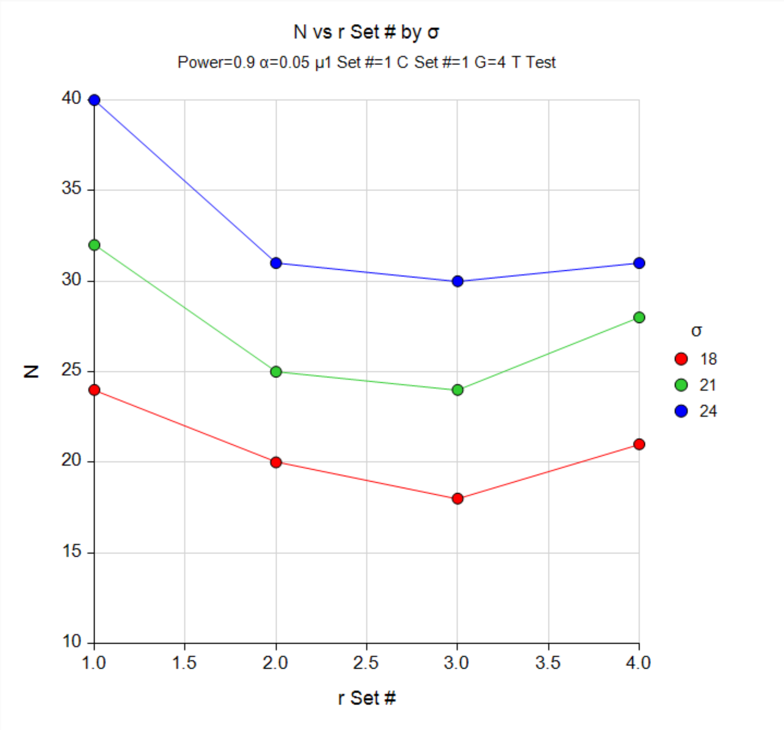
Contrast Coefficients C	Power	Sample Size Total N	Sample Size Allocation r	Sample Size Group n	Group Means|H1 Means μ1	Group Means|H1 Contrast δ1	Standard Deviation σ	Standard Error of δ's σ(δ)	Noncentrality Parameter Δ	Alpha
C(1)	0.91944	24	C1(1)	n(1)	μ1(1)	-90	18	1.414	-63.640	0.05
C(1)	0.92175	32	C1(1)	n(2)	μ1(1)	-90	21	1.225	-73.485	0.05
C(1)	0.91476	40	C1(1)	n(3)	μ1(1)	-90	24	1.095	-82.158	0.05
C(1)	0.92853	20	C2(2)	n(4)	μ1(1)	-90	18	1.369	-65.727	0.05
C(1)	0.91548	25	C2(2)	n(5)	μ1(1)	-90	21	1.225	-73.485	0.05
C(1)	0.91147	31	C2(2)	n(6)	μ1(1)	-90	24	1.092	-82.423	0.05
C(1)	0.90733	18	C3(3)	n(7)	μ1(1)	-90	18	1.414	-63.640	0.05
C(1)	0.91418	24	C3(3)	n(8)	μ1(1)	-90	21	1.225	-73.485	0.05
C(1)	0.90896	30	C3(3)	n(9)	μ1(1)	-90	24	1.095	-82.158	0.05
C(1)	0.94506	21	C4(4)	n(10)	μ1(1)	-90	18	1.323	-68.034	0.05
C(1)	0.94818	28	C4(4)	n(11)	μ1(1)	-90	21	1.146	-78.558	0.05
C(1)	0.90443	31	C4(4)	n(12)	μ1(1)	-90	24	1.106	-81.359	0.05

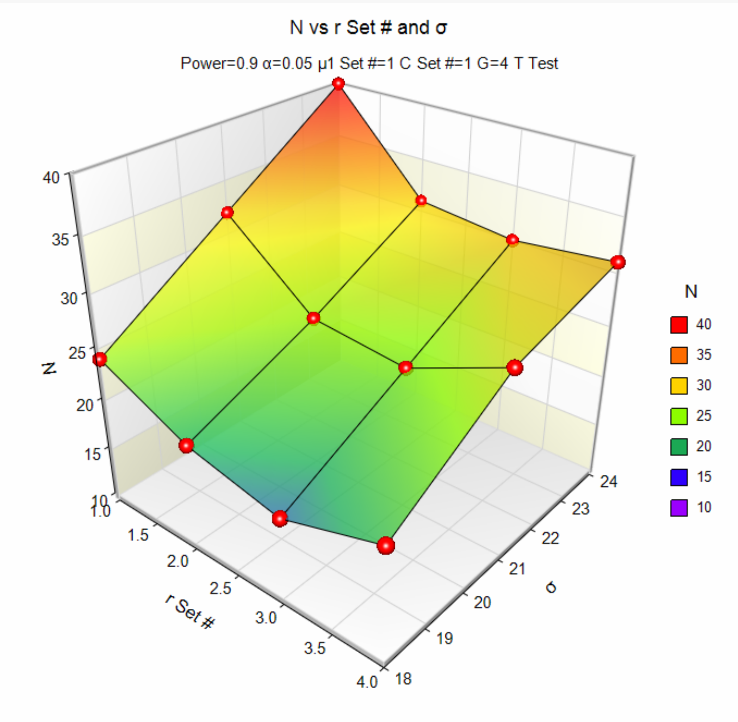
Item	Values
C(1)	-3, 1, 1, 1
C1(1)	1, 1, 1, 1
C2(2)	2, 1, 1, 1
C3(3)	3, 1, 1, 1
C4(4)	4, 1, 1, 1
n(1)	6, 6, 6, 6
n(2)	8, 8, 8, 8
n(3)	10, 10, 10, 10
n(4)	8, 4, 4, 4
n(5)	10, 5, 5, 5
n(6)	13, 6, 6, 6
n(7)	9, 3, 3, 3
n(8)	12, 4, 4, 4
n(9)	15, 5, 5, 5
n(10)	12, 3, 3, 3
n(11)	16, 4, 4, 4
n(12)	19, 4, 4, 4
μ1(1)	40, 10, 10, 10

Group Sample Size Details

n	N	Group Sample Sizes	Group Allocation Proportions
n(1)	24	6, 6, 6, 6	0.25, 0.25, 0.25, 0.25
n(2)	32	8, 8, 8, 8	0.25, 0.25, 0.25, 0.25
n(3)	40	10, 10, 10, 10	0.25, 0.25, 0.25, 0.25
n(4)	20	8, 4, 4, 4	0.4, 0.2, 0.2, 0.2
n(5)	25	10, 5, 5, 5	0.4, 0.2, 0.2, 0.2
n(6)	31	13, 6, 6, 6	0.419, 0.194, 0.194, 0.194
n(7)	18	9, 3, 3, 3	0.5, 0.167, 0.167, 0.167
n(8)	24	12, 4, 4, 4	0.5, 0.167, 0.167, 0.167
n(9)	30	15, 5, 5, 5	0.5, 0.167, 0.167, 0.167
n(10)	21	12, 3, 3, 3	0.571, 0.143, 0.143, 0.143
n(11)	28	16, 4, 4, 4	0.571, 0.143, 0.143, 0.143
n(12)	31	19, 4, 4, 4	0.613, 0.129, 0.129, 0.129

Plots





The plots show that for all values of the standard deviation, the third allocation pattern {3, 1, 1, 1} requires the minimum number of subjects. This pattern allocates 50% to the first (control) group and spreads the remaining 50% evenly among the three treatment groups. Perhaps this optimality occurs because the contrast being tested has coefficients {-3, 1, 1, 1}.