

## Chapter 247

# Cumulative Sum (CUSUM) Charts

## Introduction

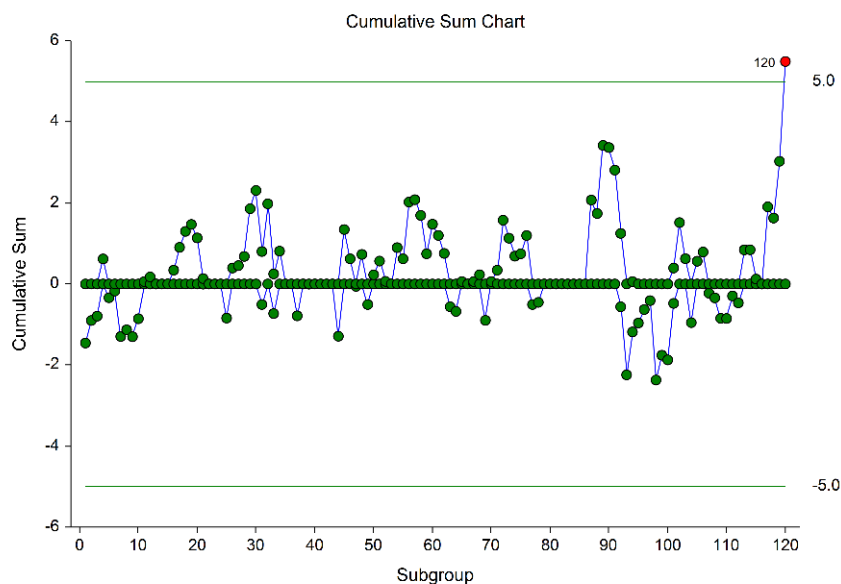
This procedure generates cumulative sum (CUSUM) control charts for. The format of the control chart is fully customizable. The data for the subgroups can be in a single column or in multiple columns. This procedure permits the defining of stages. The target value and sigma may be estimated from the data (or a subset of the data), or a target value and sigma may be entered directly. The CUSUM chart may be used for subgroup data, or for single observations at each time point. A fast initial response (head start) may be employed by the CUSUM chart, if desired.

## Cumulative Sum (CUSUM) Control Charts

The CUSUM chart is used to monitor the mean of a process based on samples taken from the process at given times (hours, shifts, days, weeks, months, etc.). The measurements of the samples at a given time constitute a subgroup. Rather than examining the mean of each subgroup independently, the CUSUM chart shows the accumulation of information of current and previous samples. For this reason, the CUSUM chart is generally better than the X-bar chart for detecting small shifts in the mean of a process.

The CUSUM chart relies on the specification of a target value and a known or reliable estimate of the standard deviation. For this reason, the CUSUM chart is better used after process control has been established.

The CUSUM chart typically signals an out-of-control process by an upward or downward drift of the cumulative sum until it crosses the boundary. An assignable cause is suspected whenever the CUSUM chart indicates an out-of-control process.



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## Other Control Charts for the Mean of a Process

The X-bar chart is the most common control chart for monitoring the process mean. The X-bar chart is usually used in Phase I monitoring, when process control is being established. The X-bar chart is useful for detecting large changes in the process mean. The CUSUM chart is based on an established target mean and a reliable value for sigma. The CUSUM chart is useful for quickly detecting small shifts in the process mean.

An alternative to the CUSUM chart is the exponentially weighted moving average (EWMA) chart. The EWMA chart has similar properties to the CUSUM chart and is also useful for detecting smaller shifts in the process mean.

When only a single response is available at each time point, then the individuals and moving range (I-MR) control charts can be used for early phase monitoring of the mean and variation. CUSUM and EWMA charts may also be used for single responses and are useful when small changes in the mean need to be detected.

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## Control Chart Formulas

Suppose we have  $k$  subgroups, each of size  $n$ . Let  $x_{ij}$  represent the measurement in the  $j^{\text{th}}$  sample of the  $i^{\text{th}}$  subgroup.

The  $i^{\text{th}}$  subgroup mean is calculated using

$$\bar{x}_i = \frac{\sum_{j=1}^n x_{ij}}{n}$$

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## Estimating the Target Value

In the CUSUM procedure, the target value may be input directly, or it may be estimated from a series of subgroups. If it is estimated from the subgroups the formula for the grand average is

$$\bar{\bar{x}} = \frac{\sum_{i=1}^k \sum_{j=1}^{n_i} x_{ij}}{\sum_{i=1}^k n_i}$$

If the subgroups are of equal size, the above equation for the grand mean reduces to

$$\bar{\bar{x}} = \frac{\sum_{i=1}^k \bar{x}_i}{k} = \frac{\bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_k}{k}$$

## Estimating Sigma – Sample Ranges

Either the range or the standard deviation of the subgroups may be used to estimate sigma, or a known (standard) sigma value may be entered directly. If the standard deviation (sigma) is to be estimated from the ranges, it is estimated as

$$\hat{\sigma} = \frac{\bar{R}}{d_2}$$

where

$$\bar{R} = \frac{\sum_{i=1}^k R_i}{k}$$

$$d_2 = \frac{E(R)}{\sigma} = \frac{\mu_R}{\sigma}$$

The calculation of  $E(R)$  requires the knowledge of the underlying distribution of the  $x_{ij}$ 's. Making the assumption that the  $x_{ij}$ 's follow the normal distribution with constant mean and variance, the values for  $d_2$  are derived through the use of numerical integration. It is important to note that the normality assumption is used and that the accuracy of this estimate requires that this assumption be valid.

When  $n$  is one, we cannot calculate  $R_i$  since it requires at least two measurements. The procedure in this case is to use the ranges of successive pairs of observations. Hence, the range of the first and second observation is computed, the range of the second and third is computed, and so on. The average of these approximate ranges is used to estimate  $\sigma$ .

## Estimating Sigma – Mean of Standard Deviations

The true standard deviation (sigma) may be input directly, or it may be estimated from the standard deviations by

$$\hat{\sigma} = \frac{\bar{s}}{c_4}$$

where

$$\bar{s} = \frac{\sum_{i=1}^k s_i}{k}$$

$$c_4 = \frac{E(s)}{\sigma} = \frac{\mu_s}{\sigma}$$

The calculation of  $E(s)$  requires the knowledge of the underlying distribution of the  $x_{ij}$ 's. Making the assumption that the  $x_{ij}$ 's follow the normal distribution with constant mean and variance, the values for  $c_4$  are obtained from

$$c_4 = \sqrt{\frac{2}{n-1} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}}$$

## Estimating Sigma – Weighted Approach

When the sample size is variable across subgroups, a weighted approach is recommended for estimating sigma (Montgomery, 2013):

$$\hat{\sigma} = \bar{s} = \left[ \frac{\sum_{i=1}^k (n_i - 1) s_i^2}{\sum_{i=1}^k n_i - k} \right]^{1/2}$$

## CUSUM Charts

Following the CUSUM procedure presented by Ryan (2011), the steps for creating a CUSUM chart may be summarized as follows:

1. Calculate the  $z_i$  using the formula

$$z_i = \frac{\bar{x}_i - \bar{\bar{x}}}{\hat{\sigma}_{\bar{x}}}$$

2. Calculate the lower and upper cumulative sums as follows

$$S_{Li} = -\max[0, (-z_i - k) + S_{Li-1}]$$

$$S_{Hi} = \max[0, (z_i - k) + S_{Hi-1}]$$

3. Plot  $S_{Hi}$  and  $S_{Li}$  on a control chart. The control limits are chosen as plus or minus  $h$ . The usual choice for  $k$  is 0.5 (for detecting one-sigma shifts in the mean) and  $h$  is typically set to 5.
4. When an out-of-control situation is detected, the corresponding sum may be left as it is, or reset to an appropriate starting value. In **NCSS**, the re-starting value may be set to zero or the fast initial restart (FIR) value of  $h/2$ .

## Data Structure

In this procedure, the data may be in either of two formats. The first data structure option is to have the data in several columns, with one subgroup per row.

### Example Dataset

S1	S2	S3	S4	S5
2	6	3	8	5
8	8	7	7	9
6	2	2	4	3
5	6	7	6	10
48	2	6	5	0
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

## Cumulative Sum (CUSUM) Charts

The second data structure option uses one column for the response data, and either a subgroup size or a second column defining the subgroups.

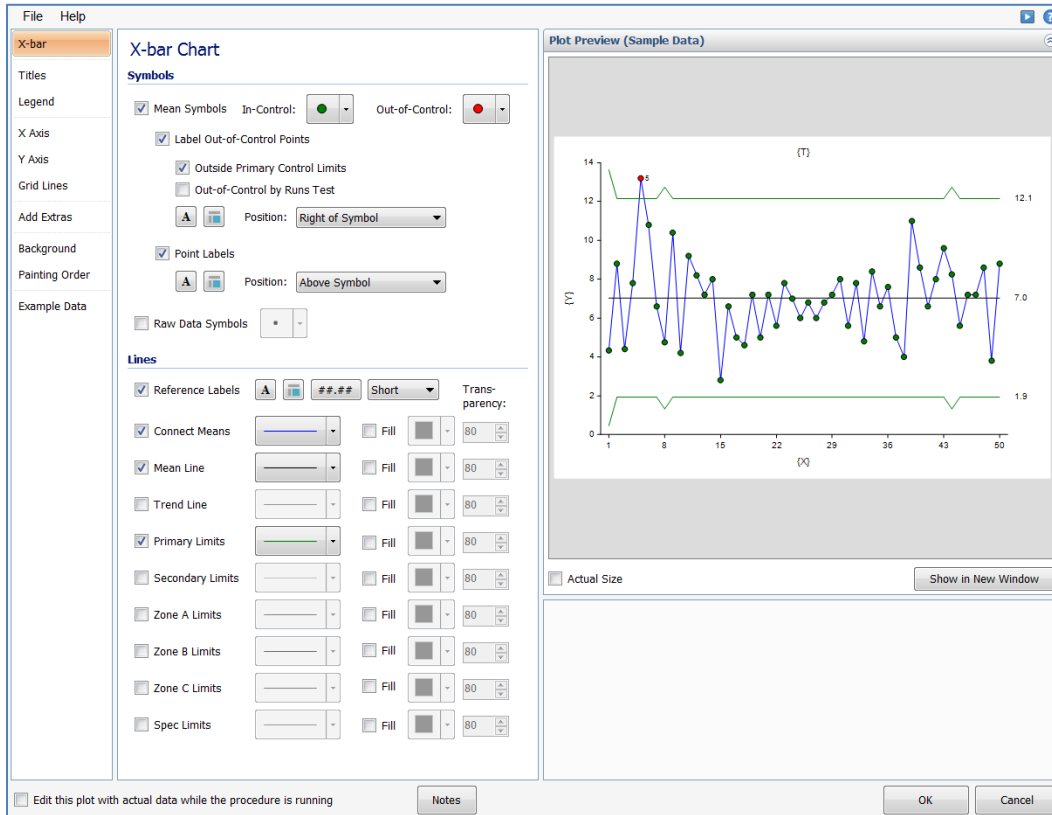
**Alternative Example Dataset**

Response	Subgroup
2	1
6	1
3	1
8	1
5	1
8	2
8	2
7	2
7	2
9	2
6	3
2	3
.	.
.	.
.	.

In the alternative example dataset, the Subgroup column is not needed if every subgroup is of size 5 and the user specifies 5 as the subgroup size. If there are missing values, the Subgroup column should be used, or the structure of the first example dataset.

# Quality Control Chart Format Window Options

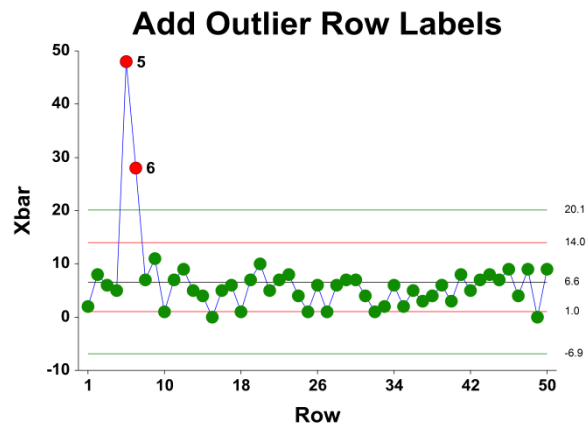
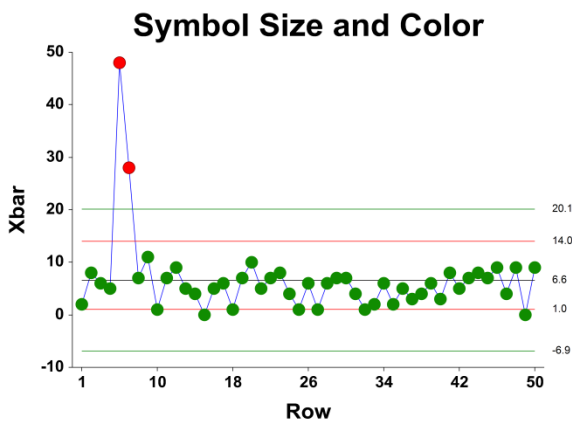
This section describes a few of the specific options available on the first tab of the control chart format window, which is displayed when a quality control chart format button is pressed. Common options, such as axes, labels, legends, and titles are documented in the Graphics Components chapter.



## [Xbar] / [Range] Chart Tab

### Symbols Section

You can modify the attributes of the symbols using the options in this section.



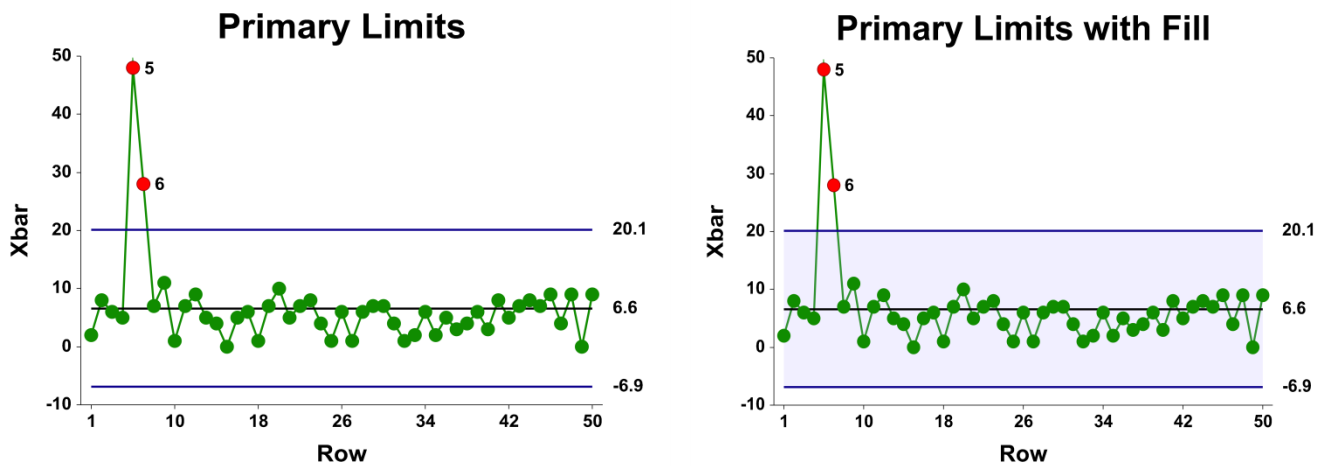
Cumulative Sum (CUSUM) Charts

A wide variety of sizes, shapes, and colors are available for the symbols. The symbols for in-control and out-of-control points are specified independently. There are additional options to label out-of-control points. The label for points outside the primary control limits is the subgroup number. The label for points that are out-of-control based on the runs test is the number of the first runs test that is signaled by this point.

The user may also specify a column of point labels on the procedure variables tab, to be used to label all or some of the points of the chart. The raw data may also be shown, based on customizable raw data symbols.

Lines Section

You can specify the format of the various lines using the options in this section. Note that when shading is desired, the fill will be to the bottom for single lines (such as the mean line), and between the lines for pairs of lines (such as primary limits).



Lines for the zones, secondary limits, and specification limits are also specified here.

Titles, Legend, Numeric Axis, Group Axis, Grid Lines, and Background Tabs

Details on setting the options in these tabs are given in the Graphics Components chapter. The legend does not show by default but can easily be included by going to the Legend tab and clicking the Show Legend checkbox.

## Example 1 – In-Control CUSUM Chart

This section presents an example of how to produce a CUSUM chart. The data represent 50 subgroups of size 5. The data used are in the QC dataset. We will analyze the variables D1 through D5 of this dataset. The target value is to be set to 67 and the sigma value is 8.

### Setup

To run this example, complete the following steps:

**1 Open the QC example dataset**

- From the File menu of the NCSS Data window, select **Open Example Data**.
- Select **QC** and click **OK**.

**2 Specify the Cumulative Sum (CUSUM) Charts procedure options**

- Find and open the **Cumulative Sum (CUSUM) Charts** procedure using the menus or the Procedure Navigator.
- The settings for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

```

Variables Tab
-----
Data Variables .....D1-D5

Target & Sigma Tab
-----
Target Value Specification.....Enter Target Value(s)
Target Value(s).....67
Sigma Specification .....Enter Sigma Value(s)
Sigma Value(s) .....8
    
```

**3 Run the procedure**

- Click the **Run** button to perform the calculations and generate the output.

### Target Value

```

Target Value for Subgroups 1 to 50
-----
Number of Subgroups: 50
-----
Target Value Specification   Value
-----
User-Specified Target Value   67
-----
    
```

This section displays the target value that is used in the CUSUM chart.



### Target Value Specification

This is the method by which the target value is obtained.

### Value

This is the target value used to create the CUSUM chart.

## Sigma Estimation

### Sigma Estimation for Subgroups 1 to 50

Number of Subgroups: 50

Sigma Specification	Value
User-Specified Sigma	8

This section shows the specification of the standard deviation used in the CUSUM chart.

### Sigma Specification

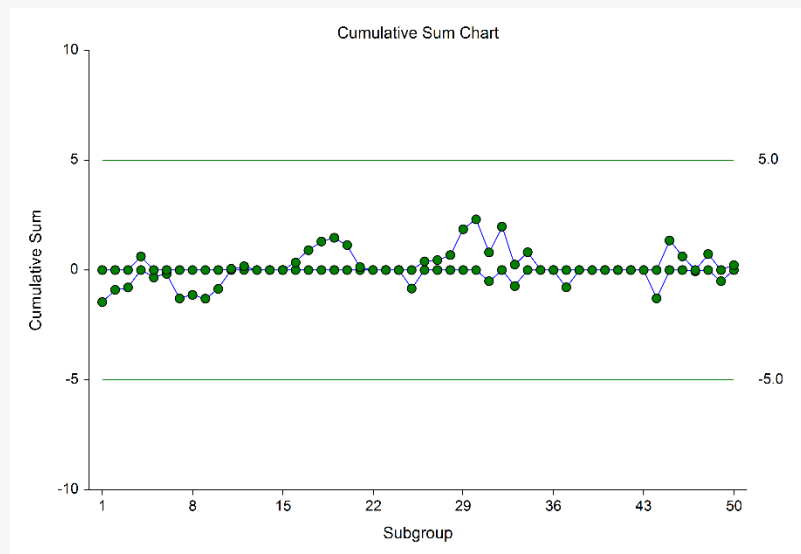
This is the method by which the sigma value is obtained.

### Value

This is the value of sigma entered by the user and is the value used to create the CUSUM chart.

## CUSUM Chart

### Charts for Subgroups 1 to 50



This CUSUM plot shows the progress of the cumulative sum across the 50 subgroups. There does not appear to be an indication of a change in the process mean.

## Example 2 – CUSUM Chart with a Change in Process Mean

This section presents a continuation of the previous example. In this example the means are monitored for an additional 100 subgroups. The data are given in the columns D1ext – D5ext of the QC dataset. In this dataset, the mean of the process changed from 67 to 70 somewhere between subgroup 110 and 120.

### Setup

To run this example, complete the following steps:

**1 Open the QC example dataset**

- From the File menu of the NCSS Data window, select **Open Example Data**.
- Select **QC** and click **OK**.

**2 Specify the Cumulative Sum (CUSUM) Charts procedure options**

- Find and open the **Cumulative Sum (CUSUM) Charts** procedure using the menus or the Procedure Navigator.
- The settings for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

```

Variables Tab
-----
Data Variables .....D1ext-D5ext

Target & Sigma Tab
-----
Target Value Specification.....Enter Target Value(s)
Target Value(s).....67
Sigma Specification .....Enter Sigma Value(s)
Sigma Value(s) .....8

CUSUM Chart Tab
-----
Restart Method .....None
    
```

**3 Run the procedure**

- Click the **Run** button to perform the calculations and generate the output.

## Target Value and Sigma Specification Sections

### Target Value for Subgroups 1 to 150

Number of Subgroups: 150

Target Value Specification	Value
User-Specified Target Value	67

### Sigma Estimation for Subgroups 1 to 150

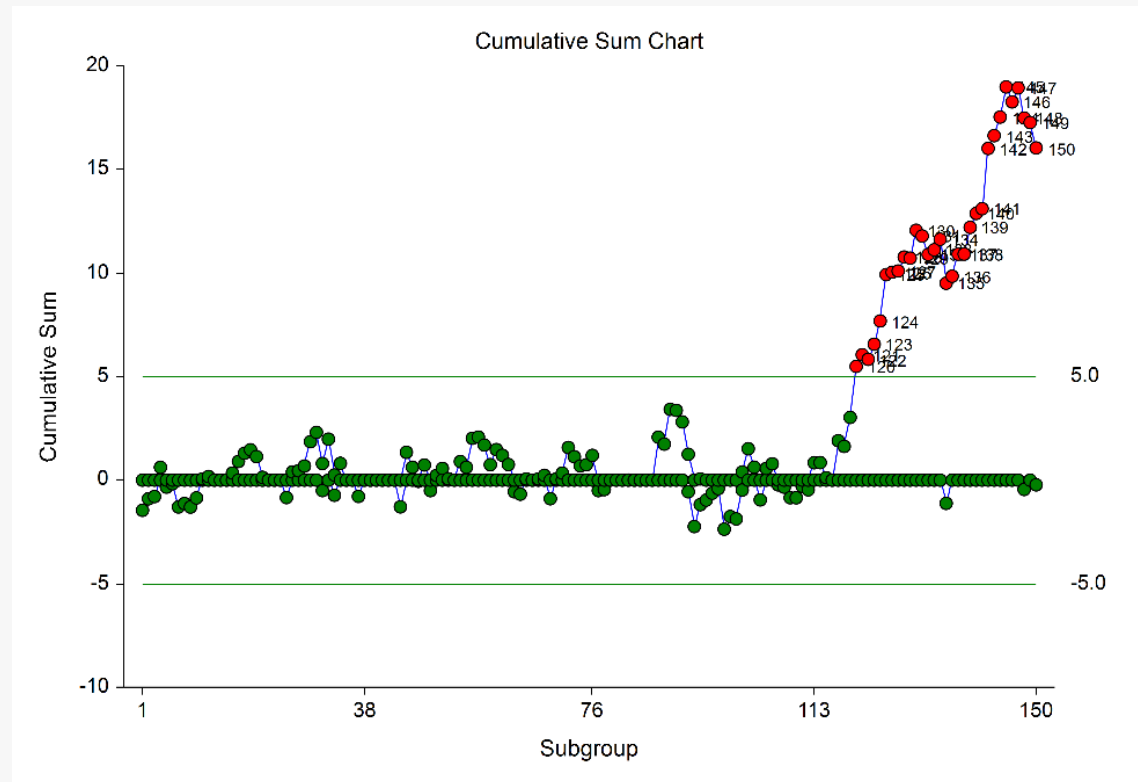
Number of Subgroups: 150

Sigma Specification	Value
User-Specified Sigma	8

Since the target value and sigma value specified are the same as those of Example 1, the results for these sections are the same as those of Example 1.

## CUSUM Chart

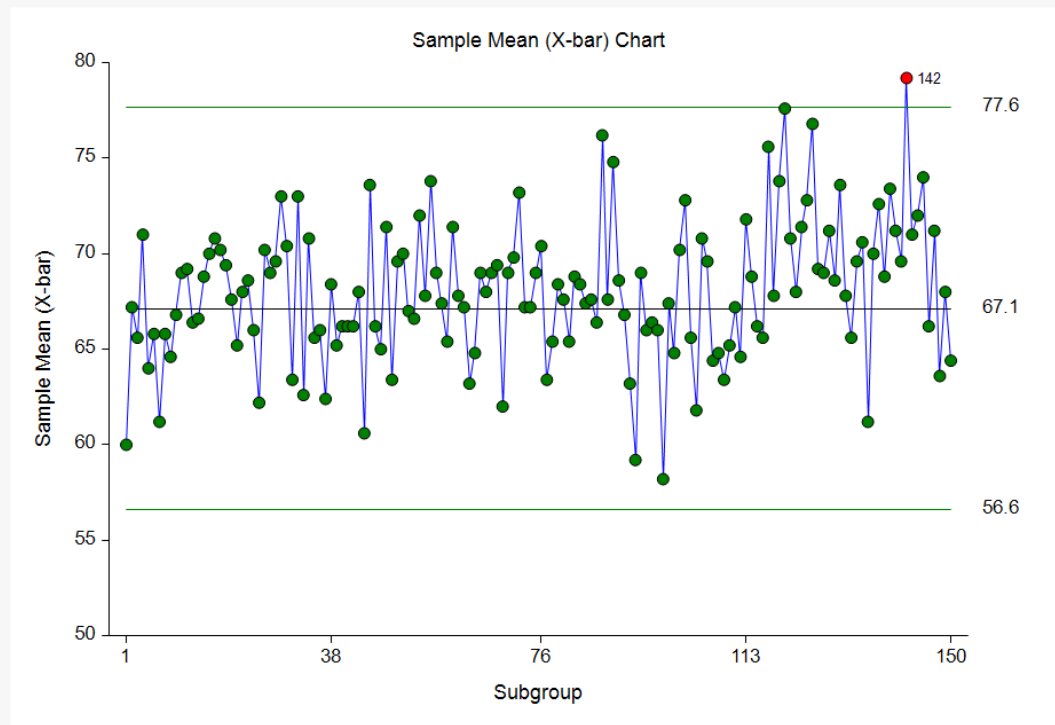
### Charts for Subgroups 1 to 150



This plot gives a clear indication of a change in the process mean somewhere shortly before subgroup 120.

## Corresponding X-bar (Mean) Chart

Charts for Subgroups 1 to 150



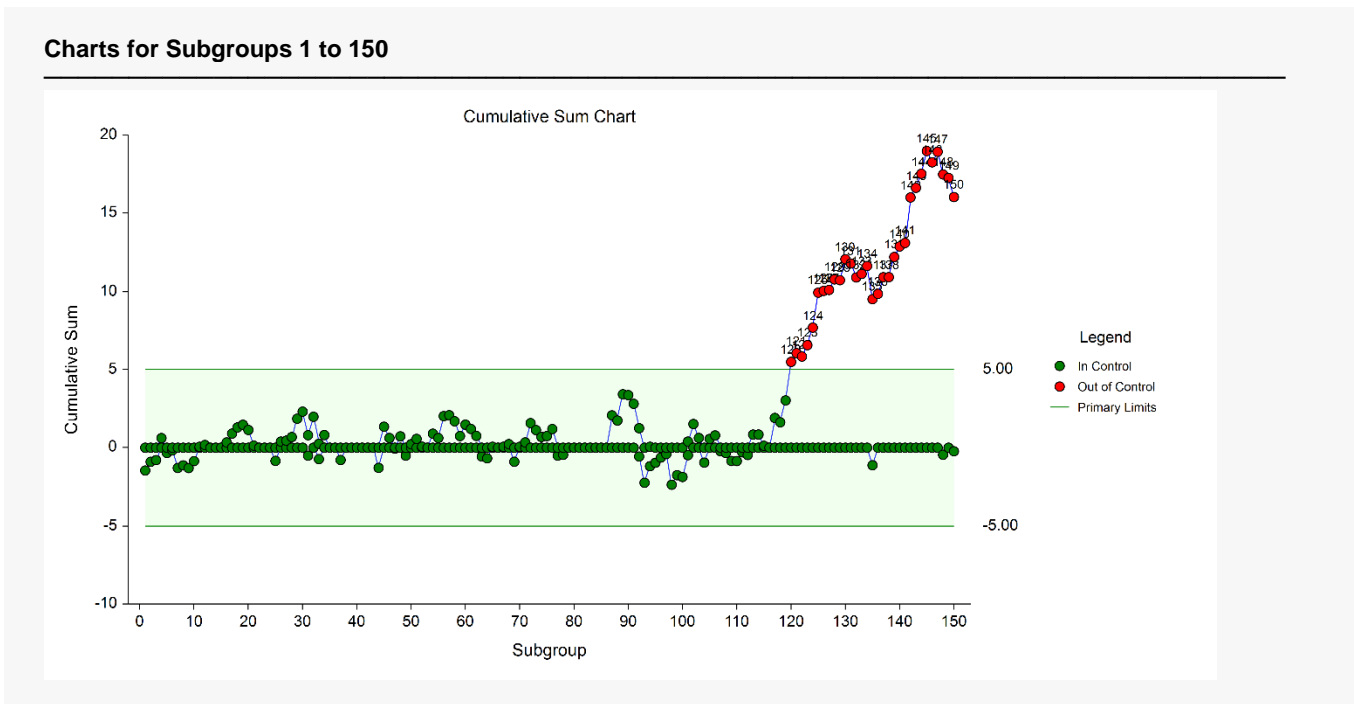
Based on the X-bar chart of the same subgroups, the change in process mean is much less apparent.

## Example 3 – CUSUM Chart with Additional Formatting

This example uses the same setup as Example 2, except that a variety of improvements are made in the plot format. These improvements are made by clicking the CUSUM Chart format button on the CUSUM Chart tab.

The settings for this example are stored in the **Example 3** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

### CUSUM Chart



As shown here, a variety of enhancements can be made to the formatting of the control charts to make the chart as easy to read as possible. The numbers above the points represent the subgroup number.