

## Chapter 295

# Correlation

---

## Introduction

The correlation coefficient, or correlation, is a unit-less measure of the relationship between two variables. The estimation of three correlation types is available in this procedure: the Pearson (product-moment) correlation, the Spearman rank correlation, and Kendall's Tau correlation. Although correlation coefficients are often reported alone, hypothesis tests and confidence intervals are also available in this procedure.

The data for calculation of the sample correlation typically comes from measurements of two variables on a number of individuals or units. The range of correlation is -1 to 1. Correlation values close to -1 indicate a strong negative relationship (high values of one variable generally indicate low values of the other). Correlation values close to 1 indicate a strong positive relationship (high values of one variable generally indicate high values of the other). Correlation values near 0 indicated little relationship among the two variables.

---

## Pearson Correlation Details

The Pearson correlation coefficient is the most common correlation measure. It is sometimes called the simple correlation coefficient, the Pearson product-moment correlation, the sample correlation coefficient, or simple linear correlation. For  $n$  measurements of two variables,  $X$  and  $Y$ , the Pearson correlation coefficient is calculated as

$$r = \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^n (X_i - \bar{X})^2 \sum_{i=1}^n (Y_i - \bar{Y})^2}}$$

---

## Confidence Intervals

The Pearson Correlation report of this procedure gives two confidence intervals for the correlation.

### Correlation Distribution Confidence Interval

Although less commonly used due to the complexity in calculation, the confidence interval for the population correlation may be computed directly from the correlation distribution. These calculations are outside the scope of this documentation, but details may be found in Pearson and Hartley (1984) and Guenther (1977). This confidence interval is preferred, even though it is not as commonly described in correlation chapters of texts. An assumption of this formulation is that the two variables are characterized by a bivariate Normal distribution.

## Correlation

## Normal Approximation Confidence Interval

An approximate (and more popular) confidence interval for the population correlation is produced by first computing a confidence interval for Fisher's Z transformation of the correlation, and then transforming back to the original correlation scale.

$$(L_Z, U_Z) = \frac{1}{2} \ln \left( \frac{1+r}{1-r} \right) \pm \frac{z_{1-\alpha/2}}{\sqrt{n-3}}$$

Convert  $L_Z$  and  $U_Z$  to the lower and upper correlation confidence limits using

$$L_\rho = \frac{e^{2L_Z} - 1}{e^{2L_Z} + 1} \quad \text{and} \quad U_\rho = \frac{e^{2U_Z} - 1}{e^{2U_Z} + 1}$$

---

## Hypothesis Tests

The formulation of the hypothesis test depends on whether the correlation is to be tested against 0, or against some other value.

### Testing $H_0: \rho = 0$

Commonly the researcher wishes to know if the correlation is different from 0, or in other words, if correlation exists. In this case, a standard  $t$ -test may be used:

$$t = \frac{r}{\sqrt{\frac{1-r^2}{n-2}}}$$

The  $t$  distribution with  $n - 2$  degrees of freedom may be used to generate one- or two-sided test  $p$ -values from this test statistic. As an aside, this test is equivalent to testing whether the linear regression slope is 0.

### Testing $H_0: \rho = \rho_0$ , where $\rho_0 \neq 0$

When testing against a correlation other than zero, either the direct distribution of the correlation may be used, or a Normal approximation based on Fisher's Z transformation of the correlation may be used.

### Correlation Distribution Test

Although less commonly used due to the complexity in calculation, the probabilities ( $p$ -values) for this hypothesis test of the correlation may be computed directly from the correlation distribution. As in the case of the confidence intervals, these calculations are outside the scope of this documentation, but details may be found in Pearson and Hartley (1984) and Guenther (1977). This test is preferred, even though it is not as commonly described in correlation chapters of texts. An assumption of this formulation is that the two variables are characterized by a bivariate Normal distribution.

## Correlation

## Normal Approximation Test

An approximate (and more popular) test statistic for the population correlation is based on Fisher's Z transformation of the correlation. This transformation has an approximate standard Normal distribution, and thus the standard Normal probabilities may be used to obtain one- or two-sided probabilities ( $p$ -values). The test statistic is

$$Z = \frac{\frac{1}{2} \ln \left( \frac{1+r}{1-r} \right) - \frac{1}{2} \ln \left( \frac{1+\rho_0}{1-\rho_0} \right)}{\sqrt{\frac{1}{n-3}}}$$

## Spearman Rank Correlation Details

The Spearman rank correlation ( $r_s$ ) is calculated by ranking the observations for each of the two variables and then computing the Pearson correlation on the ranks. When ties are encountered, the average rank is used.

## Confidence Intervals

The Fisher's Z transformation (Normal approximation) methods are used to produce confidence intervals. One adjustment is made to the variance of Z, according to the recommendation of Bonett and Wright (2000). The adjustment is to change the variance from  $1 / (n - 3)$  to  $(1 + r_s^2/2) / (n - 3)$ . It should be noted that these approximate formulas are suggested to be used only when the Spearman rank correlation is less than 0.9 and when  $n$  is 10 or greater.

## Hypothesis Tests

The formulation of the hypothesis test depends on whether the correlation is to be tested against 0, or against some other value.

### Testing $H_0: \rho = 0$

To test whether the Spearman rank correlation is significantly different from 0, the same method as that of the Pearson correlation is used, namely,

$$t = \frac{r_s}{\sqrt{\frac{1-r_s^2}{n-2}}}$$

The  $t$  distribution with  $n - 2$  degrees of freedom may be used to generate one- or two-sided test  $p$ -values from this test statistic.

## Correlation

Testing  $H_0: \rho = \rho_{s0}$ , where  $\rho_{s0} \neq 0$ 

When testing against a correlation other than zero, the Normal approximation based on Fisher's Z transformation of the correlation is used, but with the variance recommended by Bonett and Wright (2000). The test statistic is

$$Z = \frac{\frac{1}{2} \ln\left(\frac{1+r_s}{1-r_s}\right) - \frac{1}{2} \ln\left(\frac{1+\rho_{s0}}{1-\rho_{s0}}\right)}{\sqrt{\frac{1+\frac{r_s^2}{2}}{n-3}}}$$

---

## Kendall's Tau Correlation Details

Kendall's Tau correlation is also a correlation based the ranks of the observations and is another nonparametric alternative to the Pearson correlation.

Kendall's Tau correlation coefficient is calculated from a sample of  $n$  data pairs (X, Y) by first creating a variable U as the ranks of X and a variable V as the ranks of Y (ties replaced with average ranks). Kendall's Tau is then calculated from U and V using

$$\tau = \frac{2(n_C - n_D)}{\sqrt{n(n-1) - T_X} \sqrt{n(n-1) - T_Y}}$$

where

$$T_X = \sum_{i=1}^{S_X} (t_{(X)i}^2 - t_{(X)i})$$

$$T_Y = \sum_{i=1}^{S_Y} (t_{(Y)i}^2 - t_{(Y)i})$$

The parameter  $n_C$  is the total number of concordant pairs and  $n_D$  is the total number of discordant pairs. A pair of points  $(U_i, Y_i)$  and  $(U_j, Y_j)$  is said to be **concordant** when either of the following statements is true: 1.  $(U_i < U_j)$  and  $(Y_i < Y_j)$ ; 2.  $(U_i > U_j)$  and  $(Y_i > Y_j)$ . Similarly, a pair of points is said to be **discordant** when either of the following statements is true: 1.  $(U_i < U_j)$  and  $(Y_i > Y_j)$ ; 2.  $(U_i > U_j)$  and  $(Y_i < Y_j)$ . Pairs in which  $(U_i = U_j)$  or  $(Y_i = Y_j)$  are not classified as concordant or discordant and are ignored.

The value of  $t_{(X)i}$  is the number of ties in the  $i^{\text{th}}$  set of ties of the X variable. There are  $S_X$  sets of ties in the X variable. The value of  $t_{(Y)i}$  is the number of ties in the  $i^{\text{th}}$  set of ties of the Y variable. There are  $S_Y$  sets of ties in the Y variable.

## Confidence Intervals

The Fisher's Z transformation (Normal approximation) methods are used to produce confidence intervals. One adjustment is made to the variance of Z, according to the recommendation of Fieller, Hartley, and Pearson (1957). The adjustment is to change the variance from  $1 / (n - 3)$  to  $0.437 / (n - 4)$ . It should be noted that these approximate formulas are suggested to be used only when the Kendall's Tau correlation is less than 0.8.

## Hypothesis Tests

The formulation of the hypothesis test depends on whether the correlation is to be tested against 0, or against some other value.

### Testing $H_0: \tau = 0$

The statistical significance of Kendall's tau is tested using

$$Z = \frac{S + \delta}{\sigma_S}$$

where

$$S = n_c - n_d$$

$$\sigma_S^2 = \frac{(n^2 - n)(2n + 5) - T_X'' - T_Y''}{18} + \frac{T_X' T_Y'}{9(n^2 - n)(n - 2)} + \frac{T_X T_Y}{2(n^2 - n)}$$

$$T_X' = \sum_{i=1}^{S_X} (t_{(X)i}^2 - t_{(X)i}) (t_{(X)i} - 2)$$

$$T_X'' = \sum_{i=1}^{S_X} (t_{(X)i}^2 - t_{(X)i}) (2t_{(X)i} + 5)$$

$$T_Y' = \sum_{i=1}^{S_Y} (t_{(Y)i}^2 - t_{(Y)i}) (t_{(Y)i} - 2)$$

$$T_Y'' = \sum_{i=1}^{S_Y} (t_{(Y)i}^2 - t_{(Y)i}) (2t_{(Y)i} + 5)$$

$$\delta = \begin{cases} -1 & \text{if } S > 0 \\ 1 & \text{if } S < 0 \end{cases}$$

The distribution of Z is approximately normal.

## Testing $H_0: \tau = \tau_0$ , where $\tau_0 \neq 0$

When testing against a correlation other than zero, the Normal approximation based on Fisher's Z transformation of the correlation is used, but with the variance recommended by Fieller, Hartley, and Pearson (1957). The test statistic is

$$Z = \frac{\frac{1}{2} \ln \left( \frac{1 + \tau}{1 - \tau} \right) - \frac{1}{2} \ln \left( \frac{1 + \tau_0}{1 - \tau_0} \right)}{\sqrt{\frac{0.437}{n - 4}}}$$

---

## Data Structure

Correlation data are entered as two columns on the spreadsheet. A frequency column may also be used to supply counts for each row, but a frequency column is not required. An example dataset, consisting of mold spore growth and moisture levels (humidity) for 20 locations, is presented below. The data are contained in the Spore Growth dataset.

### Spore Growth Dataset

Growth	Moisture
1.8	33
3.3	57
5.4	81
6.2	80
4.5	63
1.8	38
2.2	45
4.5	63
.	.
.	.
.	.

---

## Missing Values

Rows with missing values in either of the analyzed columns are ignored.

## Example 1 – Pearson Correlation Analysis

This section presents an example of analyzing the Pearson correlation of the Spore Growth dataset. In this example, the Y axis variable will be Growth (Spore Growth), while the X axis variable will be Moisture (Humidity). A confidence interval and a test against a null correlation value of 0 will be obtained.

### Setup

To run this example, complete the following steps:

#### 1 Open the Spore Growth example dataset

- From the File menu of the NCSS Data window, select **Open Example Data**.
- Select **Spore Growth** and click **OK**.

#### 2 Specify the Correlation procedure options

- Find and open the **Correlation** procedure using the menus or the Procedure Navigator.
- The settings for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

##### Variables Tab

Y Axis Variable(s) ..... **Growth**  
X Axis Variable ..... **Moisture**

#### 3 Run the procedure

- Click the **Run** button to perform the calculations and generate the output.

### Run Summary

#### Run Summary

Parameter	Value	Parameter	Value
Y Axis Variable	Growth	Rows Processed	20
X Axis Variable	Moisture	Rows Used in Estimation	20
Frequency Variable	None	Rows with X Missing	0
Sum of Frequencies	20	Rows with Freq Missing	0

This report summarizes the variables, frequencies, and processed rows.

## Column Summary Statistics

### Column Summary Statistics

Variable	Count	Mean	Standard Deviation	Minimum	Maximum
Growth	20	3.75	1.62	1.1	7.4
Moisture	20	58.4	14.82	33	81

This report presents the count, mean, standard deviation, minimum, and maximum of the two variables.

## Pearson Correlation Confidence Intervals

### Pearson Correlation Confidence Intervals

Confidence Interval Type: Two-Sided

Pearson Correlation	Count	95% Confidence Interval Limits			
		R Distribution		Normal Approximation	
		Lower	Upper	Lower	Upper
0.8763	20	0.6991	0.947	0.7085	0.9503

The estimated Pearson correlation is 0.8763. This report presents two sets of confidence limits: one based on the exact correlation distribution, and the other based on a Normal approximation using Fisher's Z-transformation. The first is the more accurate of the two, while the second is more commonly used due to its ease in calculation. Calculation details are given in the Pearson Correlation Details section earlier in this chapter of the documentation.

## Pearson Correlation Test

### Pearson Correlation Test

Hypothesis Test:  $H_0: \rho = 0$  vs.  $H_1: \rho \neq 0$

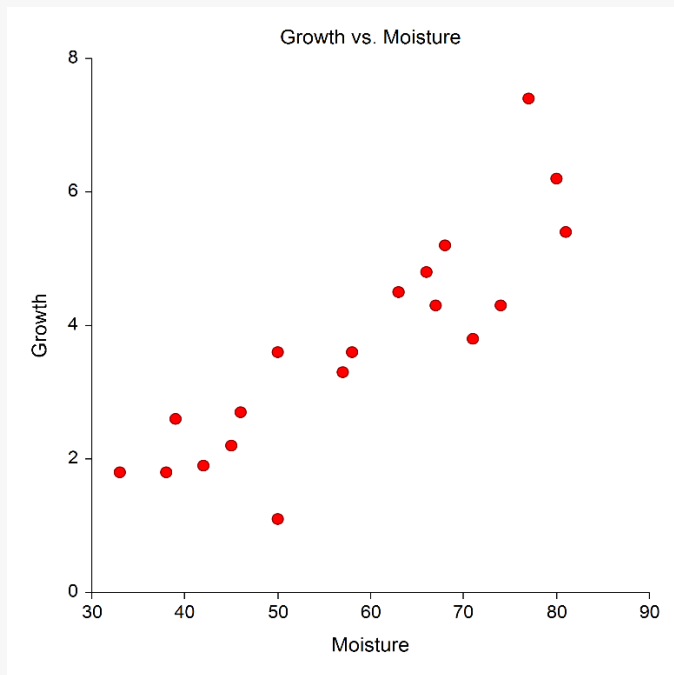
Pearson Correlation	Count	DF	T-Value	P-Value	Reject $H_0$ at $\alpha = 0.05$ ?
0.8763	20	18	7.7167	0.0000	Yes

This report gives the two-sided statistical test to determine whether the correlation is different from 0. The very small P-value indicates strong evidence that the correlation is not 0.



## Scatter Plots

### Scatter Plots



The plot shows the visual representation of the X and Y axis data. This plot may be useful for finding outliers and nonlinearities.

## Example 2 – Testing Against a Nonzero Null Hypothesis Value

This section presents an example of testing whether the correlation between spore growth and humidity is greater than 0.5. In this example, the Y axis variable will be Growth (Spore Growth), while the X axis variable will be Moisture (Humidity). The dataset used is the Spore Growth dataset.

### Setup

To run this example, complete the following steps:

#### 1 Open the Spore Growth example dataset

- From the File menu of the NCSS Data window, select **Open Example Data**.
- Select **Spore Growth** and click **OK**.

#### 2 Specify the Correlation procedure options

- Find and open the **Correlation** procedure using the menus or the Procedure Navigator.
- The settings for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

##### Variables Tab

Y Axis Variable(s) ..... **Growth**  
 X Axis Variable ..... **Moisture**

##### Reports Tab

Run Summary ..... **Unchecked**  
 Column Summary Statistics ..... **Unchecked**  
 Pearson Correlation ..... **Unchecked**  
 Confidence Intervals  
 H0:  $\rho =$  ..... **Custom**  
 Custom ..... **0.5**  
 H1: .....  **$\rho > \text{Custom Value (one-sided)}$**

##### Plots Tab

Y vs X ..... **Unchecked**

#### 3 Run the procedure

- Click the **Run** button to perform the calculations and generate the output.

## Pearson Correlation Test

### Pearson Correlation Test

Hypothesis Test:  $H_0: \rho \leq 0.5$  vs.  $H_1: \rho > 0.5$

Pearson Correlation	Count	R Distribution		Normal Approximation		
		P-Value	Reject H0 at $\alpha = 0.05$ ?	Z-Value	P-Value	Reject H0 at $\alpha = 0.05$ ?
0.8763	20	0.0007	Yes	3.3407	0.0004	Yes

This report gives two one-sided statistical tests to determine whether the correlation is statistically greater than 0.5. Both statistical tests exhibit strong evidence that the true correlation is greater than 0.5.

## Example 3 – Rank Correlation Coefficients

This section presents an example of analyzing the correlation of spore growth and humidity based on the Spearman rank correlation and Kendall's Tau. In this example, the Y axis variable will be Growth (Spore Growth), while the X axis variable will be Moisture (Humidity). The dataset used is the Spore Growth dataset.

### Setup

To run this example, complete the following steps:

#### 1 Open the Spore Growth example dataset

- From the File menu of the NCSS Data window, select **Open Example Data**.
- Select **Spore Growth** and click **OK**.

#### 2 Specify the Correlation procedure options

- Find and open the **Correlation** procedure using the menus or the Procedure Navigator.
- The settings for this example are listed below and are stored in the **Example 3** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

##### Variables Tab

Y Axis Variable(s) ..... **Growth**  
X Axis Variable ..... **Moisture**

##### Reports Tab

Run Summary ..... **Unchecked**  
Column Summary Statistics ..... **Unchecked**  
Pearson Correlation Confidence Intervals ..... **Unchecked**  
Spearman Rank Correlation ..... **Checked**  
Confidence Interval  
Kendall's Tau Correlation ..... **Checked**  
Confidence Interval  
Pearson Correlation Test ..... **Unchecked**  
Spearman Rank Correlation Test ..... **Checked**  
Kendall's Tau Correlation Test ..... **Checked**

##### Plots Tab

Y vs X ..... **Unchecked**

#### 3 Run the procedure

- Click the **Run** button to perform the calculations and generate the output.

## Spearman Rank Correlation Confidence Interval

### Spearman Rank Correlation Confidence Interval

Confidence Interval Type: Two-Sided

Spearman Correlation	Count	95% Confidence Interval Limits	
		Lower	Upper
0.8896	20	0.6954	0.9627

The confidence interval limits were computed using the Normal approximation method.

The estimated Spearman rank correlation is 0.8896. Calculation details are given in the Spearman Rank Correlation Details section earlier in this chapter of the documentation.

## Kendall's Tau Correlation Confidence Interval

### Kendall's Tau Correlation Confidence Interval

Confidence Interval Type: Two-Sided

Kendall's Tau Correlation	Count	95% Confidence Interval Limits	
		Lower	Upper
0.7326	20	0.5445	0.8506

The confidence interval limits were computed using the Normal approximation method.

The estimated Kendall's Tau correlation is 0.7326. Calculation details are given in the Kendall's Tau Correlation Details section earlier in this chapter of the documentation.

## Spearman Rank Correlation Test

### Spearman Rank Correlation Test

Hypothesis Test:  $H_0: \rho = 0$  vs.  $H_1: \rho \neq 0$

Spearman Correlation	Count	DF	T-Value	P-Value	Reject $H_0$ at $\alpha = 0.05?$
0.8896	20	18	8.2635	0.0000	Yes

This report gives the two-sided statistical test to determine whether the Spearman rank correlation is different from 0. The very small P-value indicates strong evidence that the Spearman correlation is not 0. Calculation details are given in the Spearman Rank Correlation Details section earlier in this chapter of the documentation.

## Kendall's Tau Correlation Test

### Kendall's Tau Correlation Test

Hypothesis Test:  $H_0: \rho = 0$  vs.  $H_1: \rho \neq 0$

Kendall's Tau Correlation	Count	Z-Value	P-Value	Reject H0 at $\alpha = 0.05$ ?
0.7326	20	4.4263	0.0000	Yes

The hypothesis test was conducted using the Normal approximation method.

This report gives the two-sided statistical test to determine whether the Kendall's Tau correlation is different from 0. The very small P-value indicates strong evidence that the Kendall's Tau correlation is not 0. Calculation details are given in the Kendall's Tau Correlation Details section earlier in this chapter of the documentation.