

## Chapter 513

# One Proportion – Equivalence Tests

## Introduction

This procedure computes confidence limits and equivalence hypothesis tests for a single proportion. For example, you might want confidence limits for the proportion of individuals with the common cold who took ascorbic acid (vitamin C) and recovered within twenty-four hours. You might want to test the equivalence hypothesis that the percentage of individuals with the common cold who recovered immediately after taking ascorbic acid is equivalent to the standard proportion of 50%.

Exact results, based on the binomial distribution, are calculated. Approximate results, based on the normal approximation to the binomial distribution are also given. In the case of a single proportion, the exact results are preferable to the approximate results and should always be used. The approximate results are available in the software because they are commonly presented in elementary statistical texts.

This procedure accepts data entered as summary counts (number of “successes” and sample size), but can also tabulate data from columns in the database.

## The Binomial Model

Binomial data must exhibit the following four conditions:

1. The response can take on only one of two possible values. This is a binary response variable.
2. The response is observed a known number of times. Each replication is called a Bernoulli trial. The number of replications is labeled  $n$ . The number of responses out of the  $n$  total that exhibit the outcome of interest is labeled  $X$ . Thus  $X$  takes on the possible values 0, 1, 2, ...,  $n$ .
3. The probability that a particular outcome (a success) occurs is constant for each trial. This probability is labeled  $P$ .
4. The trials are independent. The outcome of one trial does not influence the outcome of the any other trial.

The binomial probability,  $b(X; n, P)$ , is calculated using:

$$b(X; n, P) = \binom{n}{X} P^X (1 - P)^{n-X}$$

where

$$\binom{n}{X} = \frac{n!}{X!(n - X)!}$$

The estimate of  $P$  from a sample is labeled  $p$  and is estimated using:

$$p = \frac{X}{n}$$

The label  $\hat{p}$  is often used in place of  $p$  in practice.

## Confidence Limits

Using a mathematical relationship (see Ostle(1988), page 110) between the  $F$  distribution and the cumulative binomial distribution, the lower and upper confidence limits of a 100  $(1 - \alpha)$ % confidence interval are given by:

$$LCL = \frac{XF_{[\alpha/2],[2X,2(n-X+1)]}}{(n - X + 1) + XF_{[\alpha/2],[2X,2(n-X+1)]}}$$

and

$$UCL = \frac{(X + 1)F_{[1-\alpha/2],[2(X+1),2(n-X)]}}{(n - X) + (X + 1)F_{[1-\alpha/2],[2(X+1),2(n-X)]}}$$

Note that although these limits are based on direct calculation of the binomial distribution, they are only “exact” for a few values of alpha. Otherwise, these limits are conservative (wider than necessary).

These limits may be approximated using the normal approximation to the binomial as

$$p \pm z_{\alpha/2} \sqrt{\frac{p(1-p)}{n}}$$

If a correction for continuity is added, the above formula becomes

$$p \pm \left( z_{\alpha/2} \sqrt{\frac{p(1-p)}{n}} + \frac{1}{2n} \right)$$

Although these two approximate confidence intervals are found in many elementary statistics books, they are not recommended in general. For example, Newcombe (1998) made a comparative study of seven confidence interval techniques and these methods came in last.

Instead, Newcombe (1998) recommended the Wilson Score confidence interval method because of its performance. The Wilson Score confidence interval is calculated using

$$\frac{(2np + z_{\alpha/2}^2) \pm z_{\alpha/2} \sqrt{z_{\alpha/2}^2 + 4np(1-p)}}{2(n + z_{\alpha/2}^2)}$$

## Equivalence Hypothesis Tests using TOST (Two One-Sided Tests)

Schuirmann’s (1987) two one-sided tests (TOST) approach is used to test equivalence. The equivalence test essentially reverses the roles of the null and alternative hypothesis. Assume that  $P$  represent the population proportion of the response,  $S$  is a standard reference proportion, and  $M$  is the so-called *margin of equivalence*. The null and alternative hypotheses are

$$\begin{aligned} H_0: P < S - M \text{ or } P > S + M \\ H_1: S - M < P < S + M \end{aligned}$$

The null hypothesis is made up of two simple one-sided hypotheses:

$$\begin{aligned} H_{0_1}: P < S - M \\ H_{0_2}: P > S + M \end{aligned}$$

If both of these one-sided tests are rejected, we conclude  $H_1$  that the response is equivalent to the standard proportion (their difference is confined within a small margin). Schuirmann showed that if we want the alpha level of the equivalence test to be  $\alpha$ , then each of the one-sided tests should be  $\alpha$  as well (not  $\alpha/2$  as you might expect). The probability level (p-value) of the equivalence test is equal to the maximum of the probability levels of the two one-sided tests. These tests are conducted using the standard formulas for the one-sample proportions test.

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The exact  $p$ -values for each of these situations may be computed as follows:

1.  $P(|\tilde{p} - P_0| \geq |p - P_0|)$ , where  $\tilde{p}$  represents all possible values of  $p$ . This probability is calculated using the binomial distribution.
2.  $\sum_{r=0}^X b(r; n, p)$
3.  $\sum_{r=X}^n b(r; n, p)$

The simple large-sample  $z$ -test is based on the test statistic

$$z = \frac{p - P_0}{\sqrt{\frac{P_0(1 - P_0)}{n}}}$$

Two approximations to the exact  $p$ -values are also available. One uses  $P_0$  and the other uses  $p$  in the calculation of the standard error. The first approximation uses  $P_0$  in the calculation of the standard error:

$$z_c = \frac{X + 0.5 - nP_0}{\sqrt{nP_0(1 - P_0)}}$$

if  $X < nP_0$ , or

$$z_c = \frac{X - 0.5 - nP_0}{\sqrt{nP_0(1 - P_0)}}$$

if  $X > nP_0$ .

The second approximation uses  $p$  in the calculation of the standard error:

$$z_c = \frac{X + 0.5 - nP_0}{\sqrt{np(1 - p)}}$$

if  $X < nP_0$ , or

$$z_c = \frac{X - 0.5 - nP_0}{\sqrt{np(1 - p)}}$$

if  $X > nP_0$ .

These  $z$ -values are used to calculate probabilities using the standard normal probability distribution.

## Data Structure

The procedure accepts data entered as summary counts (number of “successes” ( $X$ ) and sample size ( $n$ )), but can also tabulate data from columns in the database. A separate report is generated for each outcome variable entered.

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## Procedure Options

This section describes the options available in this procedure.

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### Data Tab

This panel specifies the variables or data that will be used in the analysis.

#### Type of Data Input

There are two formats that may be used to load the data into this procedure.

- **Enter the Number of “Successes” (X) and the Sample Size (n)**

For this selection, you enter the number of “successes” ( $X$ ) (items with the attribute of interest) and the total sample size ( $n$ ).

- **Tabulate “Successes” from a Categorical Outcome Variable in the Database**

Use this option when you have raw data that must be tabulated. You will be asked to select a column in the database that contains the categorical outcome values (such as “0” and “1” or “No” and “Yes”). The data in the outcome variable will be read and summarized.

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### Summary Data

#### Number of “Successes” (X)

Enter the number of individuals in the sample for which the result was a “Success”. For a given scenario, a “Success” might indicate “Attribute of Interest”, “Positive Response”, “Disease”, or something else. This value must be a whole number between 0 and the Sample Size. Some of the formulas are unable to give meaningful results if the Number of Success is equal to 0 or the sample size.

#### Sample Size (n)

Enter the number of individuals or experimental units in the sample. This number must be a whole number greater than zero.

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### Database Input – Categorical Outcome Variable(s) to be Tabulated

#### Outcome Variable(s)

Specify one or more categorical variables used to define the outcome or response. If more than one variable is specified, a separate analysis is performed for each variable. This procedure analyzes only one outcome, determined by the “Success” Value below.

#### “Success” Value

Specify the value in the output variable that is to be considered a “success” and tabulated. This is the value of interest on which the proportions and analyses will be based. The entry is not case-sensitive.

#### Values and Value Labels

You may enter either the value or the value label (if specified) for the outcome of interest.

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### Database Input – Frequency Variable

#### Frequency Variable

Specify an optional column containing the number of observations (cases) represented by each row. If this option is left blank, each row of the dataset is assumed to represent one observation.

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### Database Input – Break Variables

#### Break Variable(s)

Enter up to 8 categorical break variables. The values in these variables are used to break the output up into separate reports and plots. A separate set of reports is generated for each unique value (or unique combination of values if multiple break variables are specified). The break variables will be applied in the order that they are listed here.

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### Equivalence Test Options

#### Equivalence Bounds

This option allows the user to specify how the equivalence bounds are created. The choices are

- **Symmetric**

The equivalence bounds are symmetric about the Standard Proportion. In this case you enter a single margin value and the equivalence bounds are computed as

$$\text{Upper Equivalence Bound} = (\text{Standard Proportion}) + (\text{Equivalence Margin})$$

$$\text{Lower Equivalence Bound} = (\text{Standard Proportion}) - (\text{Equivalence Margin})$$

- **Asymmetric**

The upper and lower equivalence bounds are entered separately and a Standard Proportion is not required.

#### Standard Proportion (Shown when Equivalence Bounds = “Symmetric”)

This is the known standard or reference proportion against which the sample proportion will be compared.

#### Equivalence Margin (Shown when Equivalence Bounds = “Symmetric”)

Enter the desired value for the margin of equivalence. The equivalence bounds are computed as

$$\text{Upper Equivalence Bound} = (\text{Standard Value}) + (\text{Equivalence Margin})$$

$$\text{Lower Equivalence Bound} = (\text{Standard Value}) - (\text{Equivalence Margin})$$

#### Range

This value should be positive.

#### Upper and Lower Equivalence Bounds (Shown when Equivalence Bounds = “Asymmetric”)

Enter the values for the upper and lower equivalence bounds. The bounds should be greater than 0 and less than 1, and the Upper Equivalence Bound should be greater than the Lower Equivalence Bound.

## Reports Tab

This panel specifies the reports to be produced in the analysis.

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### Data Report

#### Data Report

Check this box to obtain a summary of the data entered, as well as the sample proportion.

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### Confidence Intervals

#### Confidence Level

This confidence level is used for all the confidence interval reports of this procedure. Typical confidence levels are 90%, 95%, and 99%, with 95% being the most common.

#### Confidence Interval Report

Check each box to obtain the corresponding confidence interval report. The details and formulas for each report are given in the documentation above.

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### Tests

#### Alpha

Enter the value of alpha to be used for all hypothesis tests in this procedure. The probability level ( $p$ -value) is compared to alpha to determine whether to reject the null hypothesis.

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## Report Options Tab

The following options control the format of the reports.

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### Report Options

#### Variable Names

Specify whether to use variable names, variable labels, or both to label output reports. In this discussion, the variables are the columns of the data table.

- **Names**  
Variable names are the column headings that appear on the data table. They may be modified by clicking the Column Info button on the Data window or by clicking the right mouse button while the mouse is pointing to the column heading.
- **Labels**  
This refers to the optional labels that may be specified for each column. Clicking the Column Info button on the Data window allows you to enter them.
- **Both**  
Both the variable names and labels are displayed.

#### Comments

1. Most reports are formatted to receive about 12 characters for variable names.
2. Variable Names cannot contain blanks or math symbols (like + - \* / . ,), but variable labels can.

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### Value Labels

Value Labels are used to make reports more legible by assigning meaningful labels to numbers and codes.

The options are

- **Data Values**  
All data are displayed in their original format, regardless of whether a value label has been set or not.
- **Value Labels**  
All values of variables that have a value label variable designated are converted to their corresponding value label when they are output. This does not modify their value during computation.
- **Both**  
Both data value and value label are displayed.

### Example

A variable named GENDER (used as a grouping variable) contains 1's and 2's. By specifying a value label for GENDER, the report can display "Male" instead of 1 and "Female" instead of 2. This option specifies whether (and how) to use the value labels.

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## Decimal Places

### Item Decimal Places

These decimal options allow the user to specify the number of decimal places for items in the output. Your choice here will not affect calculations; it will only affect the format of the output.

#### Auto

If one of the "Auto" options is selected, the ending zero digits are not shown. For example, if "Auto (0 to 7)" is chosen,

0.0500 is displayed as 0.05

1.314583689 is displayed as 1.314584

The output formatting system is not designed to accommodate "Auto (0 to 13)", and if chosen, this will likely lead to lines that run on to a second line. This option is included, however, for the rare case when a very large number of decimals is needed.

## Example 1 – Equivalence Test of One Proportion using Summary Data

This section presents an example of how to run an equivalence test of summary data in which  $n$  is 100,  $X$  is 55, the standard proportion,  $S$ , is 0.50, and the equivalence margin,  $M$ , is set to 0.10.

You may follow along here by making the appropriate entries or load the completed template **Example 1** by clicking on Open Example Template from the File menu of the One Proportion - Equivalence Tests window.

### 1 Open the One Proportion - Equivalence Tests window.

- Using the Analysis menu or the Procedure Navigator, find and select the **One Proportion - Equivalence Tests** procedure.
- On the menus, select **File**, then **New Template**. This will fill the procedure with the default template.

### 2 Specify the data and test.

- Select the **Data** tab.
- For **Type of Data Input** select **Enter the Number of “Successes” (X) and the Sample Size (n)**.
- In the **Number of “Successes” (X)** box, enter **55**.
- In the **Sample Size (n)** box, enter **100**.
- For **Equivalence Bounds** select **Symmetric**.
- For **Standard Proportion** enter **0.50**.
- For **Equivalence Margin** enter **0.10**.

### 3 Specify the reports.

- Select the **Reports** tab.
- Check all **confidence interval** reports.
- Check all **tests**.

### 4 Run the procedure.

- From the Run menu, select **Run Procedure**. Alternatively, just click the green **Run** button.

## Data Report

Data Report		
Number of "Successes" (X)	Sample Size (n)	Sample Proportion (p)
55	100	0.55

This report documents the values that were entered.

## Confidence Limits Section

Exact Confidence Interval of P				95% C. I. of P	
Number of "Successes" (X)	Sample Size (n)	Sample Proportion (p)	Z*	Lower Limit	Upper Limit
55	100	0.55	1.95996	0.4472802	0.6496798



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## Simple Z Confidence Interval of P

Number of "Successes" (X)	Sample Size (n)	Sample Proportion (p)	Z*	95% C. I. of P	
				Lower Limit	Upper Limit
55	100	0.55	1.95996	0.452493	0.647507

## Simple Z Confidence Interval of P with Continuity Correction

Number of "Successes" (X)	Sample Size (n)	Sample Proportion (p)	Z*	95% C. I. of P	
				Lower Limit	Upper Limit
55	100	0.55	1.95996	0.447493	0.6525069

## Wilson Score Confidence Interval of P

Number of "Successes" (X)	Sample Size (n)	Sample Proportion (p)	Z*	95% C. I. of P	
				Lower Limit	Upper Limit
55	100	0.55	1.95996	0.452446	0.6438546

The Wilson score interval is the interval recommended in Newcombe (1998). The limits are based on the formulas that were presented earlier.

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**Hypothesis Test Section**

## Exact Test for Equivalence using TOST (Two One-Sided Tests)

Equivalence Hypothesis:  $0.5 - 0.1 < P < 0.5 + 0.1$

Test	Alternative Hypothesis	Sample Proportion	Prob Level	Reject H0 at $\alpha = 0.05?$
Lower Boundary	$P > 0.4$	0.55	0.00171	Yes
Upper Boundary	$P < 0.6$	0.55	0.17890	No
Equivalence	$0.4 < P < 0.6$		0.17890	No

## Simple Z-Test for Equivalence using TOST (Two One-Sided Tests)

Equivalence Hypothesis:  $0.5 - 0.1 < P < 0.5 + 0.1$

Test	Alternative Hypothesis	Sample Proportion	Z-Statistic	Prob Level	Reject H0 at $\alpha = 0.05?$
Lower Boundary	$P > 0.4$	0.55	3.06190	0.00110	Yes
Upper Boundary	$P < 0.6$	0.55	-1.02060	0.15372	No
Equivalence	$0.4 < P < 0.6$			0.15372	No

## Z Approximation Test using P0 for Equivalence using TOST (Two One-Sided Tests)

Equivalence Hypothesis:  $0.5 - 0.1 < P < 0.5 + 0.1$

Test	Alternative Hypothesis	Sample Proportion	Z-Statistic	Prob Level	Reject H0 at $\alpha = 0.05?$
Lower Boundary	$P > 0.4$	0.55	2.95980	0.00154	Yes
Upper Boundary	$P < 0.6$	0.55	-0.91860	0.17916	No
Equivalence	$0.4 < P < 0.6$			0.17916	No

## Z Approximation Test using p for Equivalence using TOST (Two One-Sided Tests)

Equivalence Hypothesis:  $0.5 - 0.1 < P < 0.5 + 0.1$

Test	Alternative Hypothesis	Sample Proportion	Z-Statistic	Prob Level	Reject H0 at $\alpha = 0.05?$
Lower Boundary	$P > 0.4$	0.55	2.91460	0.00178	Yes
Upper Boundary	$P < 0.6$	0.55	-0.90450	0.18286	No
Equivalence	$0.4 < P < 0.6$			0.18286	No

The formulas for these tests were shown earlier. The test fail to conclude equivalence.

## Example 2 – Equivalence Test of One Proportion using Raw Data

This section presents an example of how to run an analysis of raw data that may have been used to generate the summary data in example 1. The data for this example are found in the **Prop1** dataset.

You may follow along here by making the appropriate entries or load the completed template **Example 2** by clicking on Open Example Template from the File menu of the One Proportion - Equivalence Tests window.

### 1 Open the Prop1 dataset.

- From the File menu of the NCSS Data window, select **Open Example Data**.
- Click on the file **Prop1.NCSS**.
- Click **Open**.

### 2 Open the One Proportion - Equivalence Tests window.

- Using the Analysis menu or the Procedure Navigator, find and select the **One Proportion - Equivalence Tests** procedure.
- On the menus, select **File**, then **New Template**. This will fill the procedure with the default template.

### 3 Specify the variables.

- Select the **Data** tab.
- For **Type of Data Input** select **Tabulate “Successes” from a Categorical Outcome Variable in the Database**.
- Double-click in the **Outcome Variable(s)** text box. This will bring up the variable selection window.
- Select **Response** from the list of variables and then click **Ok**. “Response” will appear in the Outcome Variable(s) box.
- For **“Success” Value**, enter **1**. (Alternatively, since this dataset contains value labels, you could enter **Yes** for **“Success” Value** since the label for a value of 1 is “Yes”. See settings template **Example 2b**.)
- For **Equivalence Bounds** select **Symmetric**.
- For **Standard Proportion** enter **0.50**.
- For **Non-Inferiority Margin** enter **0.10**.

### 4 Specify the reports.

- Select the **Reports** tab.
- Check all **confidence interval** reports.
- Check all **tests**.

### 5 Run the procedure.

- From the Run menu, select **Run Procedure**. Alternatively, just click the green **Run** button.

## Output

### Data Report for Response

Number of "Successes" (X)	Sample Size (n)	Sample Proportion (p)
55	100	0.55

(Report continues with results exactly as in Example 1)

The results are exactly the same as they were in example 1.