# Chapter 468

# Spectral Analysis

## Introduction

This program calculates and displays the periodogram and spectrum of a time series. This is sometimes known as harmonic analysis or the frequency approach to time series analysis.

Suppose we believe that a time series, $X_t$, contains a periodic (cyclic) component. A natural model of the periodic component would be

$$X_t = R\cos(ft + d) + e_t$$

where

*R* is the amplitude of variation. Normally, the cosine varies between -1 and 1. Hence, if *R* is 6, then the term would vary between -6 and 6. The impact of the amplitude is in the size (height or magnitude) of the wave. The length of the wave is not influenced by the amplitude.

*f* is the frequency of periodic variation, measured in number of radians per unit time. This is the 'frequency' scale of the plots. If we divide $2\pi$ by *f*, we get the corresponding *wavelength*. This is the 'wavelength' scale of the plots. The impact of the frequency is to change the length of a cycle. As *f* increases, the length of the cycle decreases. A model with *f* = 2 would have a cycle length equal to one-half the cycle length of a model with *f* = 1.

*d* is the phase. Changing the phase causes a shift in the beginning of the cycle.

$e_t$ is the random error (noise) of the series about the period component.

*t* is the time period number. Usually, *t=1, 2, 3, ..., N.*

Since *cos(ft+d) = cos(ft) cos(d) - sin(ft) sin(d)*, this model may be written in the alternative form

$$X_t = a\cos(ft) + b\sin(ft) + e_t$$

where *a = R cos(d)* and *b = -R sin(d).*

This model is a multiple regression model with two independent variables. In this case, the independent variables are *X1 = cos(ft)* and *X2 = sin(ft).* The regression coefficients are *B1 = a* and *B2 = b*. In practice, the variation in a time series may be modeled as the sum of several different individual waves occurring at different frequencies.

The generalization of this model to the sum of *k* frequencies may be written symbolically as

$$X_t = \sum_{j=1}^{k} R_j \cos\left(f_j t + d_j\right) + e_t$$

or, using the alternative form, as

$$X_t = \sum_{j=1}^{k} a_j \cos\left(f_j t\right) + \sum_{j=1}^{k} b_j \sin\left(f_j t\right) + e_t$$

Note that if the $f_j$ were known constants, and we let $W_{tr} = \cos\left(f_r t\right)$ and $Z_{ts} = \sin\left(f_s t\right)$, then this could be rewritten in the usual multiple regression form:

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$$X_t = \sum_{j=1}^{k} a_j W_{tj} + \sum_{j=1}^{k} b_j Z_{tj} + e_t$$

where the *a's* and the *b's* are the regression coefficients to be estimated. This is an example of a harmonic regression.

Fourier analysis is the study of approximating functions using the sum of sine and cosine terms. This sum is called the Fourier series representation of the function. Spectral analysis is identical to Fourier analysis except that instead of approximating a function, the sum of sine and cosine terms approximates a time series that includes a random component. Note that the coefficients (the *a's* and *b's*) may be estimated using multiple regression.

One question that arises is how to select the frequencies. The highest frequency that can be fit to the data is $\pi$. The lowest is one cycle for the whole length of series, which amounts to a frequency of $2\pi / N$ (*N* is the length of the series). Hence, one popular choice of frequencies is to select the *N/2* frequencies given by

$$f_k = 2\pi k / N, \quad (k = 1, 2, \cdots, N/2)$$

The k[th] frequency is often referred to as the k[th] harmonic.

This set of frequencies is particularly popular when working by hand because it results in certain simplifications due to well-known trigonometric identities. However, there is nothing in nature that says that a series will follow these rather than some other set. That is why the program lets you specify a range of frequencies.

In the analysis of variance, we study the partitioning of the total variation (sum of squares) given by

$$SST = \sum_{t=1}^{N} \left( X_t - \overline{X} \right)^2$$

into the sum of squares for factor A, factor B, etc. Similarly, in spectral analysis we are interested in partitioning the total sum of squares into amounts associated with each frequency. It turns out that the sum of squares for a particular frequency, $SS_k$, is given by

$$SS_k = \frac{N}{2}\left( a_k^2 + b_k^2 \right)$$

If we regard $SS_k$ as the portion of the total sum of squares accounted for by frequencies in the range

$$f_k \pm \frac{\pi}{N},$$

we can draw a histogram so that the area of each bar is proportional $SS_k$. The height of the histogram would be

$$I\left( f_k \right) = \frac{N}{4\pi}\left( a_k^2 + b_k^2 \right)$$

The plot of *I(f)* versus *f* is called the *periodogram*.

This definition of the periodogram equates the total sum of squares to the area under the periodogram. *I(f)* may be calculated directly from the data as

$$I\left( f_k \right) = \frac{\left[ \sum X_t \cos\left( 2\pi k t / N \right) \right]^2 + \left[ \sum X_t \sin\left( 2\pi k t / N \right) \right]^2}{N\pi}$$

The periodogram is sometimes calculated using the fast Fourier transform (FFT). This method is not used in this program for three reasons. First, the increase in speed of the FFT is not significant until *N* is greater than one thousand. For series of the length we normally anticipate for our users, the FFT would provide little speed improvement.

Second, when using the FFT, the length of the series (*N*) must be a power of 2 (2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, etc.). If *N* is not a power of 2, then enough zeros must be added to bring the length of the series to the next

power of 2. Suppose the length of a particular series was 260. You would need to add 252 zeros to bring the length to 512. This could dramatically distort your results. (FFT users use various “windows” or “filters” to remove the effect of these zeros. Since we do not pad with zeros, we do not need these filters.)

Third, we can calculate the periodogram for any set of frequencies, not just the set given above. This is very useful when you want to investigate a particular range of frequencies.

The sample periodogram has been shown to have some poor statistical properties. Recently, techniques for spectral analysis have improved on the periodogram by smoothing it. The smoothed periodogram is an estimate of the *power spectral density* or simply the *spectral density* of the series. The smoothing used in this program is simply an *m-term* moving average of the periodogram. The value of *m* is specified as the Smoothing Length option. Practitioners suggest that a value of *m* near *N/40* is reasonable. A large value of *m* may make the graph too smooth while a value too small may include spurious peaks.

Spectral analysis offers an interesting addition to other methods of time series analysis. For those who wish to find more out about it, we strongly recommend the book by C. Chatfield (1984). It offers a thorough, readable treatment of a difficult, but useful, subject.

# Data Structure

The data are entered in a single variable.

# Missing Values

When missing values are found in the series, they are either replaced or omitted. The replacement value is the average of the nearest observation in the future and in the past or the nearest non-missing value in the past.

If you do not feel that this is a valid estimate of the missing value, you should manually enter a more reasonable estimate before using the algorithm. These missing value replacement methods are particularly poor for seasonal data. We recommend that you replace missing values manually before using the algorithm.

# Procedure Options

This section describes the options available in this procedure.

## Variables Tab

Specify the variable on which to run the analysis.

### Time Series Variable

#### Time Series Variable

Specify the variable on which to run the analysis.

#### Use Logarithms

Specifies that the log (base 10) transformation should be applied to the values of the variable.

#### Missing Values

Choose how missing (blank) values are processed.

The algorithm used in this procedure cannot tolerate missing values since each row is assumed to represent the next point in a time sequence. Hence, when missing values are found, they must be removed either by imputation (filling in with a reasonable value) or by skipping the row and pretending it does not exist.

Whenever possible, we recommend that you replace missing values manually.

Here are the available options.

**Average the Adjacent Values**

Replace the missing value with the average of the nearest values in the future (below) and in the past (above).

**Carry the Previous Value Forward**

Replace the missing value with the first non-missing value immediately above (previous) this value.

**Omit Row from Calculations**

Ignore the row in all calculations. Analyze the data as if the row was not on the database.

## Data Adjustment Options

### Remove Mean

Checking this option indicates that the series average should be subtracted from the data. This is almost always done.

### Remove Trend

Checking this option indicates that the least squares trend line should be subtracted from the data. This is sometimes done, although differencing is usually used to remove trends instead.

### Regular Differencing

This option lets you designate whether the original series, the first differences, or the second differences are analyzed. The first difference series, *W*, is calculated using the formula:

$$W_t = X_t - X_{t-1}$$

which may be written using the backshift operator, B, as:

$$W_t = (1 - B) X_t$$

The second difference series, *Z*, is the first difference of the *W* series. The formula is:

$$Z_t = W_t - W_{t-1}$$

which may be written using the backshift operator, B, as:

$$Z_t = (1 - B)^2 X_t$$

### Seasonal Differencing

This option lets you designate whether the original series, the first seasonal differences, or the second seasonal differences are analyzed. Assuming the number of seasons is *s*, the first seasonal difference series, *W*, is calculated using the formula:

$$W_t = X_t - X_{t-s}$$

which may be written using the backshift operator, B, as:

$$W_t = (1 - B^s) X_t$$

The second seasonal difference series, *Z*, is the first seasonal difference of the *W* series. The formula is:

$$Z_t = W_t - W_{t-s}$$

which may be written using the backshift operator, B, as:

$$Z_t = \left(1 - B^s\right)^2 X_t$$

## Seasonality Options

### Seasons

Specify the number of seasons, *s*, in the series. Use '4' for quarterly data or '12' for monthly data. Note that this option is used only when seasonal differencing is used.

# Reports Tab

The following options control which reports are displayed.

## Select Reports

### Fourier Report

This option specifies whether the indicated report is displayed.

## Periodogram / Spectrum Calculation Options

### Number of Frequencies

Specify the number of frequencies that are calculated and displayed. This controls the resolution of the periodogram and spectrum. The frequencies are equi-spaced between the minimum and maximum wavelengths.

### Smoothing Length

The spectral density function is a moving average of the periodogram. This option specifies the value of *m*, the number of periodogram terms averaged.

### Minimum Wavelength

The minimum wavelength value to be used in calculating and displaying the periodogram and spectral density.

### Maximum Wavelength

The maximum wavelength value to be used in calculating and displaying the periodogram and spectral density. The maximum value possible is *N*, the sample size.

## Report Options

### Precision

Specify the precision of numbers in the report. Single precision will display seven-place accuracy, while the double precision will display thirteen-place accuracy. Note that all reports are formatted for single precision only.

### Variable Names

Specify whether to use variable names or (the longer) variable labels in report headings.

# Plots Tab

This section controls the inclusion and the settings of the plots.

## Select Plots

### Data Plot - Spectrum

Each of these options specifies whether the indicated plot is displayed. Click the plot format button to change the plot settings.

## Horizontal Axis Variable if there are Missing or Filtered Values

### Horizontal Variable

This option controls the spacing on the horizontal axis when missing or filtered values occur.

Your choices are

**Actual Row Number**

Use the actual row number of each row from the dataset along the horizontal axis.

**Sequence Number**

Use the sequence (relative row) number formed by ignoring any missing or filtered values.

# Example 1 – Spectral Analysis

This section presents an example of how to do a spectral analysis of a time series. The Spots variable in the Sunspot dataset will be used.

You may follow along here by making the appropriate entries or load the completed template **Example 1** by clicking on Open Example Template from the File menu of the Spectral Analysis window.

**1 Open the Sunspot dataset.**

- From the File menu of the NCSS Data window, select **Open Example Data**.
- Click on the file **Sunspot.NCSS**.
- Click **Open**.

**2 Open the Spectral Analysis window.**

- Using the Analysis menu or the Procedure Navigator, find and select the **Spectral Analysis** procedure.
- On the menus, select **File**, then **New Template**. This will fill the procedure with the default template.

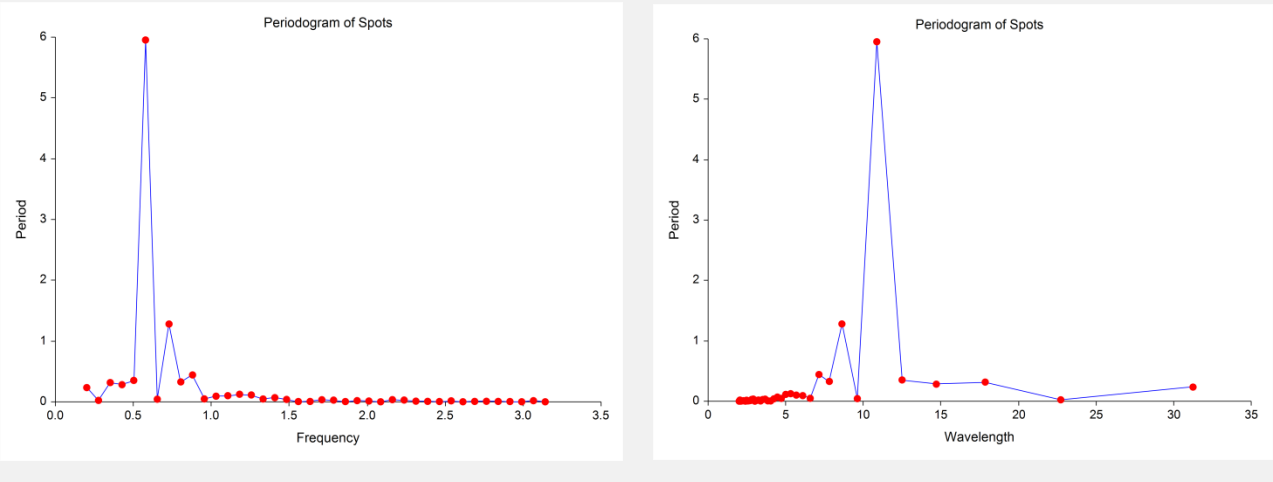
**3 Specify the variables.**

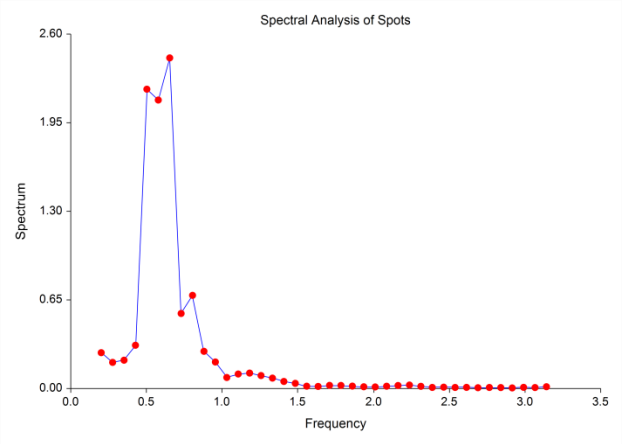
- On the Spectral Analysis window, select the **Variables tab**.
- Double-click in the **Time Series Variable** box. This will bring up the variable selection window.
- Select **Spots** from the list of variables and then click **Ok**.

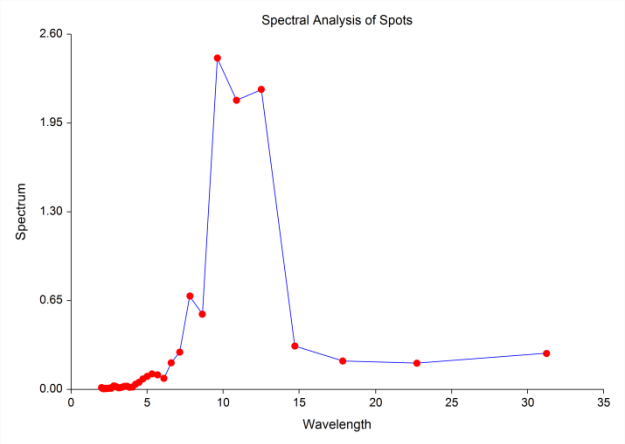
**4 Run the procedure.**

- From the Run menu, select **Run Procedure**. Alternatively, just click the green Run button.

## Fourier Plot Section

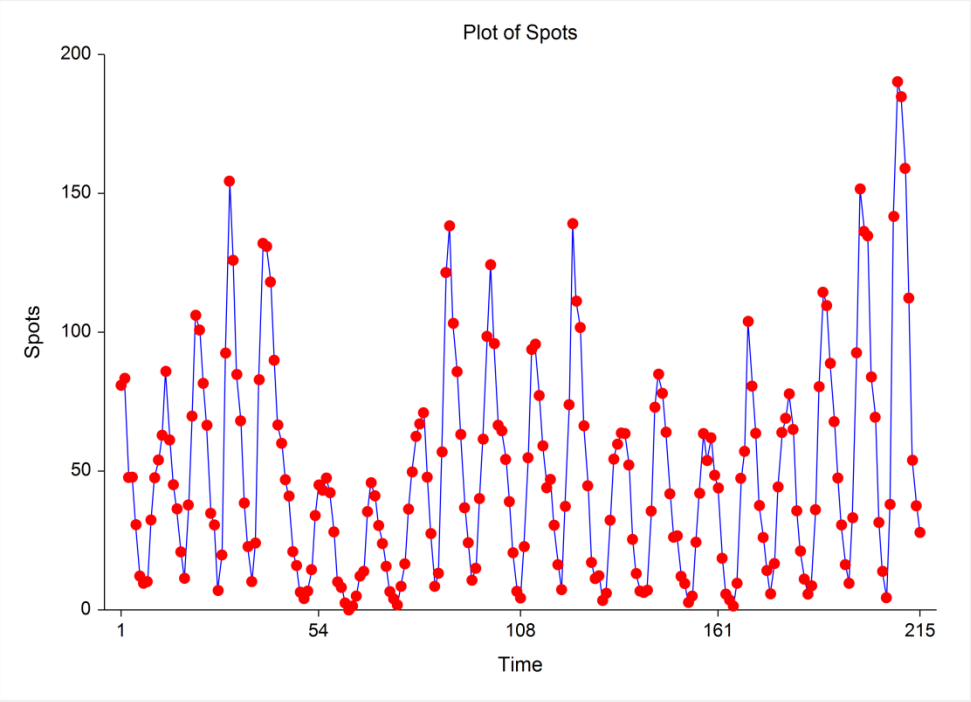





This section displays the periodogram and the spectrum plots with the frequency scale and the wavelength scale. Remember that the wavelength is in terms of the number of observations.

## Data Plot Section




This section displays a plot of the data values.

# Fourier Analysis Section

**Fourier Analysis of SPOTS (0,0,12,1,0)**

| Frequency | Wavelength | Period | Cosine(a's) | Sine(b's) | Sprectrum |
|---|---|---|---|---|---|
| 0.2010619 | 31.25 | 3200767 | -75.89938 | -425.8151 | 3590384 |
| 0.2764601 | 22.72727 | 324968.6 | -13.48533 | 137.1568 | 2618775 |
| 0.3518584 | 17.85714 | 4330590 | 92.67065 | -494.4972 | 2837492 |
| 0.4272566 | 14.70588 | 3856917 | 33.71187 | -473.5963 | 4333876 |
| 0.5026549 | 12.5 | 4814120 | 298.3864 | -438.5685 | 2.997943E+07 |
| . | . | . | . | . | . |
| . | . | . | . | . | . |
| . | . | . | . | . | . |

This section shows the values of the various components of the spectral analysis. The numbers in parentheses, (d,D,s,M,T), are defined as follows:

**d** is the regular differencing order.

**D** is the seasonal differencing order.

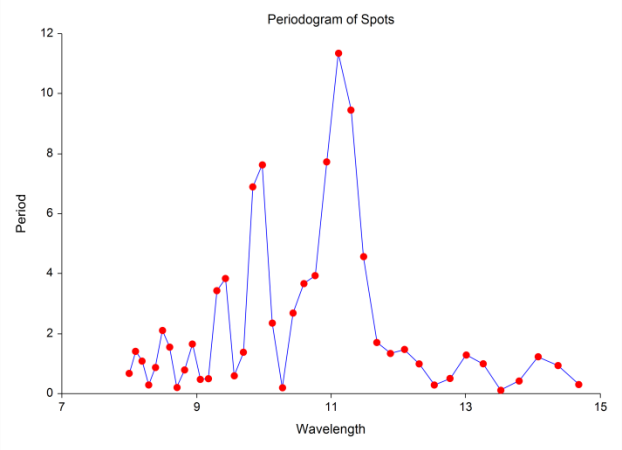
**s** is the number of seasons (ignored if D is 0).

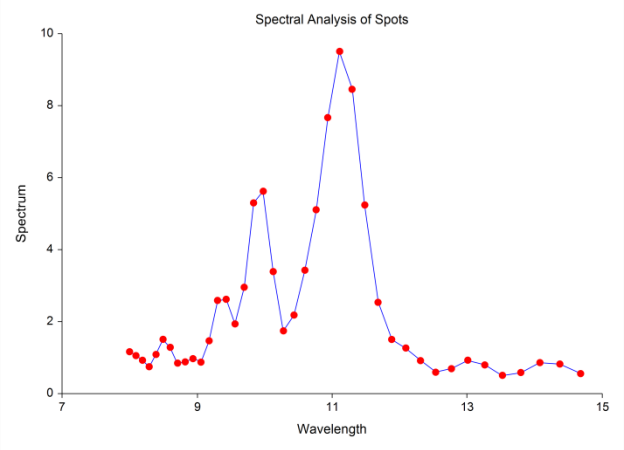
**M** is 1 if the mean is subtracted, 0 otherwise.

**T** is 1 if the trend is subtracted, 0 otherwise.

# Fourier Plot Section

To complete this example, we rerun the analysis with the minimum wavelength set to 8 and the maximum wavelength set to 15. This appears to be portion of the periodogram and spectrum that show the most promise. Doing this produces the following wavelength plots.




Now we can see the famous sunspot cycle of just over eleven years.