Chapter 592

Analysis of Covariance (ANCOVA) Contrasts

Introduction

This procedure provides power and sample size for studies using a one-way design in which the data are to be analyzed using analysis of covariance with tests formed using contrasts of the means. The calculations use the more accurate results of Shieh (2017).

A common task in research is to compare the means of two or more populations (groups). We might want to compare the income level of two regions, the nitrogen content of three lakes, or the effectiveness of four drugs. Analysis of Covariance (ANCOVA) is an extension of the one-way analysis of variance model that adds quantitative variables (covariates). When used, it is assumed that their inclusion will reduce the size of the error variance and thus increase the power of the design.

The usual F-test tests the hypothesis that all means are equal versus the alternative that at least one mean is different from the rest. Often, a more specific alternative is desired. For example, you might want to test whether the treatment means are different from the control mean, the low dose is different from the high dose, a linear trend exists across dose levels, and so on. These questions can be tested using specific contrasts.

Assumptions

Using the *F* test requires certain assumptions. One reason for the popularity of the *F* test is its robustness in the face of assumption violation. However, if an assumption is not even approximately met, the significance levels and the power of the *F* test are invalidated. Unfortunately, in practice it often happens that several assumptions are not met. This makes matters even worse. Hence, steps should be taken to check the assumptions before important decisions are made.

The assumptions of the one-way analysis of variance are:

1. The data are continuous (not discrete).
2. The data follow the normal probability distribution. Each group is normally distributed about the group mean.
3. The variances within the groups are equal.
4. The groups are independent. There is no relationship among the individuals in one group as compared to another.
5. Each group is a simple random sample from its population. Each individual in the population has an equal probability of being selected in the sample.

Additional assumptions are needed for an analysis of covariance:

1. The covariates have a linear relationship with the response variable.
2. The slopes of these linear relationships between the covariate and the response variable are approximately equal across all groups.

Technical Details for ANCOVA Contrasts

This procedure replaces an older version of ANCOVA analysis of contrasts with an updated version based on Shieh (2017). This article included the results of extensive simulation studies that showed that the new algorithm presented is much more accurate than the older algorithm used in previous versions. Note that this previous version is still available as a legacy procedure.

Suppose *G* groups each have a normal distribution and with covariate-adjusted means $\mu_1, \mu_2, \ldots, \mu_G$. Also suppose that each observation consists of a response measurement, *Y*, and one or more covariate measurements: X_1, X_2, …, X_p. The older algorithm required the assumption that the values of these covariates were fixed, known constants. Shieh's algorithm makes a more reasonable assumption that the covariates are random samples from a multivariate normal population which do not have to be known during the planning phase.

Further suppose that samples of $n_1, n_2, \ldots, n_G$ observations will be obtained from each of *G* groups. The multiple regression equation relating *Y* to the *X's* within the i^{th} group is

$$Y_i \;=\; \beta_{0i} + \beta_1 X_1 + \beta_2 X_2 + \;\cdots\; + \beta_p X_p + \varepsilon$$

The $\beta's$ are the regression coefficients or slopes. Analysis of covariance assumes that, except for the intercepts, β_{0i}, the slopes are equal across all groups. Thus, the difference between the means of any two groups is equal to the difference between their intercepts. Note that Shieh (2017) mentions that this requirement may be relaxed to some degree.

Let σ^2 denote the common variance of all groups ignoring the covariates and σ_ε^2 the within-group variance after considering the covariates. These values are related according to the formula

$$\sigma_\varepsilon^2 = \sigma^2(1-\rho^2)$$

where ρ^2 is the coefficient of multiple determination (estimated by R^2).

Suppose *G* groups each have a normal distribution and equal means ($\mu_1 = \mu_2 = \cdots = \mu_G$). Let $n_1 = n_2 = \cdots = n_G$ denote the number of subjects in each group and let *N* denote the total sample size of all groups. Let σ denote the common standard deviation of all groups.

A *comparison* is a weighted average of the means, in which the weights may be negative. When the weights sum to zero, the comparison is called a *contrast*. This procedure provides results for contrasts used in an ANCOVA design.

Suppose you want to test whether the contrast

$$\delta = \sum_{i=1}^{G} c_i \mu_i$$

is significantly different from zero. Here $c_1, c_2, \ldots, c_G$ are the contrast coefficients.

Note that $\sum_{i=1}^{G} c_i = 0$ is required. Also, to make different contrasts comparable, Kirk (2013) suggests that $\sum_{i=1}^{G} |c_i| = 2$.

Note that the effect size (and noncentrality parameter) associated with this contrast may be defined as

$$\Delta = \sum_{i=1}^{G} \frac{c_i \mu_i}{\sigma_\varepsilon}$$

Test Statistic

A test statistic for testing $\delta = 0$ is given by

$$T^* = \frac{\hat{\delta} - 0}{\widehat{\omega}}$$

where

$$\widehat{\omega} = \hat{\sigma}_\varepsilon \sqrt{V}$$

$$\hat{\sigma}_\varepsilon = SSE/v$$

$$SSE = \sum_{i=1}^{G} \sum_{j=1}^{n_i} \left(Y_{ij} - \bar{Y}_i\right)^2 - S_{XY}^T S_{XX}^{-1} S_{XY}$$

$$N_T = \sum_{i=1}^{G} n_i$$

$$v = N_T - G - p$$

$$V = a(1 + bF^*)$$

$$a = \sum_{i=1}^{G} \frac{c_i^2}{n_i}$$

$$b = \frac{p}{N_T - G - p + 1}$$

$$F^* = \left(\sum_{i=1}^{G} c_i \bar{X}_i\right)^T S_{XX}^{-1} \left(\sum_{i=1}^{G} c_i \bar{X}_i\right) / ab$$

The test statistic T^* is distributed as a *t* distribution with *v* degrees of freedom. Hence, the null hypothesis, $H_0\colon \delta = 0$, is rejected if $|T^*| > t_{v,1-\alpha/2}$.

Power Calculations for Contrasts

The power of the test is given in Shieh (2017) as follows.

$$\text{Power}(\Delta_{\text{TB}}) = E_B\left[P\left\{\left|t_{v,\Delta}\right| > t_{v,1-\alpha/2}\right\}\right]$$

where

$$\Delta = \frac{\delta}{\sqrt{a\sigma^2}}$$

$$\Delta_{TB} = \Delta\sqrt{B^*}$$

$$B^* = 1/(1 + bF^*)$$

$$T^*|B^* \sim t'_{v,\Delta_{TB}}$$

$$B^* \sim Beta\left\{\frac{v+1}{2}, \frac{p}{2}\right\}$$

The expectation E_B is taken with respect to the distribution of B^*.

Example 1 – Finding Power

An experiment is being designed to compare the means of four groups using a contrast test from an ANCOVA at a significance level of 0.05. There will be three covariates.

The first group is a control group. The other three groups will have slightly different treatments applied. The researchers are mainly interested in whether the three treatment groups are different from the control group. Hence, they want to test the contrast represented by the coefficients -1, 0.333, 0.333, 0.334. Treatment means of 20, 11, 10, and 12 represent clinically important group differences. Note that this set of means has an overall range of 20 – 10 = 10, which was the main requirement of the researchers.

Previous studies have had standard deviations of residuals between 12 and 18. To better understand the relationship between power and sample size, the researcher wants to compute the power for several group sample sizes between 10 and 40. The sample sizes will be equal across all groups.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For .... **Power**
Alpha.. **0.05**
G (Number of Groups) **4**
Group Allocation Input Type .... **Equal to ni (Sample Size per Group)**
ni (Sample Size Per Group) **10 20 30 40**
μ1 Input Type.. **Enter μ1 (Group Means|H1)**
μ1 (Group Means|H1) **20 11 10 12**
Contrast Coefficients Input Type.................... **List of Contrast Coefficients**
Contrast Coefficients..................................... **-1 0.333 0.333 0.334**
Standard Deviation Input Type .... **Enter σe (SD of Residuals)**
σe (SD of Residuals) **12 15 18**
Number of Covariates................................... **3**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Reports

Numeric Results

Solve For: Power
Number of Groups: 4
Number of Covariates: 3

Contrast Coefficients C	Power	Sample Size Total N	Sample Size Group ni	Group Means|H1 Means μ1	Group Means|H1 Contrast δ	Standard Deviation of Residuals σe	Alpha
C(1)	0.48025	40	10	μ1(1)	-8.999	12	0.05
C(1)	0.33346	40	10	μ1(1)	-8.999	15	0.05
C(1)	0.24709	40	10	μ1(1)	-8.999	18	0.05
C(1)	0.80179	80	20	μ1(1)	-8.999	12	0.05
C(1)	0.61300	80	20	μ1(1)	-8.999	15	0.05
C(1)	0.46535	80	20	μ1(1)	-8.999	18	0.05
C(1)	0.93551	120	30	μ1(1)	-8.999	12	0.05
C(1)	0.79499	120	30	μ1(1)	-8.999	15	0.05
C(1)	0.64068	120	30	μ1(1)	-8.999	18	0.05
C(1)	0.98089	160	40	μ1(1)	-8.999	12	0.05
C(1)	0.89820	160	40	μ1(1)	-8.999	15	0.05
C(1)	0.76844	160	40	μ1(1)	-8.999	18	0.05

The number of numerical integration intervals used in calculations was 2000.

Item	Values
C(1)	-1, 0.333, 0.333, 0.334
μ1(1)	20, 11, 10, 12

C The name of the set containing the contrast coefficients. The only restriction is that the sum of the coefficients must be zero and δ ≠ 0.
Power The probability of rejecting a false null hypothesis when the alternative hypothesis is true.
N The total number of subjects in the study.
ni The number of items sampled from each group.
μ1 The Group Means | H1 is the set name and number of the group means under the alternative hypothesis. This is the set of means at which the power is calculated.
δ The Contrast Among Means is the result of applying the contrast to the group means. It is calculated using δ = C'μ1.
σe The standard deviation of the residuals from a regression of the response on both groups and covariates.
Alpha The probability of rejecting a true null hypothesis.

Group Sample Size Details

n	N	Group Sample Sizes	Group Allocation Proportions
n(1)	40	10, 10, 10, 10	0.25, 0.25, 0.25, 0.25
n(2)	80	20, 20, 20, 20	0.25, 0.25, 0.25, 0.25
n(3)	120	30, 30, 30, 30	0.25, 0.25, 0.25, 0.25
n(4)	160	40, 40, 40, 40	0.25, 0.25, 0.25, 0.25

Summary Statements

An analysis of covariance design with 4 groups and 3 covariates will be used to test whether the contrast of the means (δ) is different from 0 (H0: δ = 0 versus H1: δ ≠ 0). The comparison will be made using a two-sided (ANCOVA) t-test using the contrast coefficients -1, 0.333, 0.333, 0.334, with a Type I error rate (α) of 0.05. The group means under the null hypothesis are assumed to be equal. The standard deviation of the residuals is assumed to be 12. To detect group means of 20, 11, 10, 12 (or a contrast of means value of -8.999), with group subject counts of 10, 10, 10, 10 (totaling 40 subjects), the power is 0.48025.

Dropout-Inflated Sample Size

Dropout Rate	Sample Size N	Dropout-Inflated Enrollment Sample Size N'	Expected Number of Dropouts D
20%	40	50	10
20%	80	100	20
20%	120	150	30
20%	160	200	40

Dropout Rate The percentage of subjects (or items) that are expected to be lost at random during the course of the study and for whom no response data will be collected (i.e., will be treated as "missing"). Abbreviated as DR.
N The evaluable sample size at which power is computed (as entered by the user). If N subjects are evaluated out of the N' subjects that are enrolled in the study, the design will achieve the stated power.
N' The total number of subjects that should be enrolled in the study in order to obtain N evaluable subjects, based on the assumed dropout rate. N' is calculated by inflating N using the formula N' = N / (1 - DR), with N' always rounded up. (See Julious, S.A. (2010) pages 52-53, or Chow, S.C., Shao, J., Wang, H., and Lokhnygina, Y. (2018) pages 32-33.)
D The expected number of dropouts. D = N' - N.

Dropout Summary Statements

Anticipating a 20% dropout rate, 50 subjects should be enrolled to obtain a final sample size of 40 subjects.

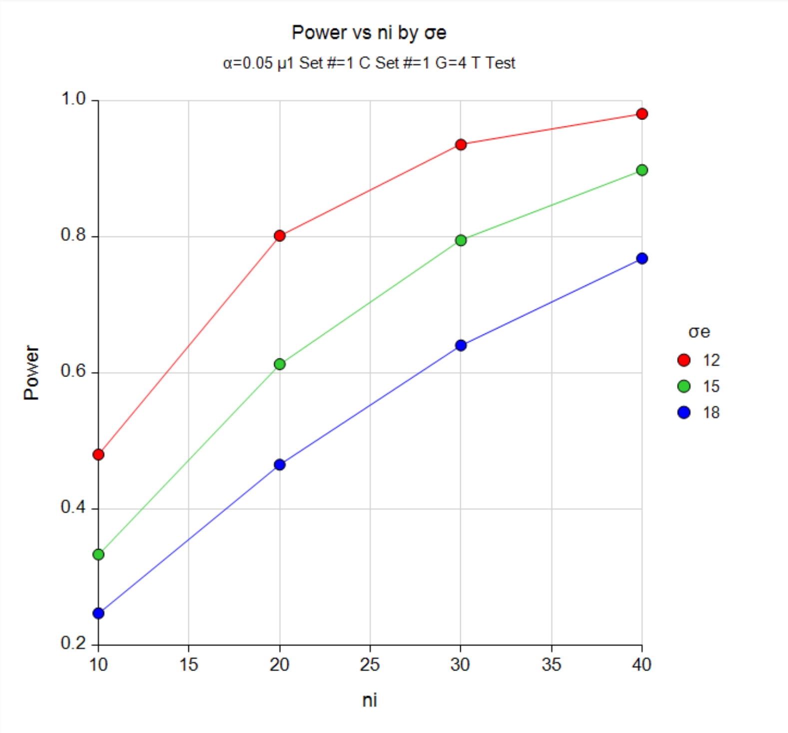
References

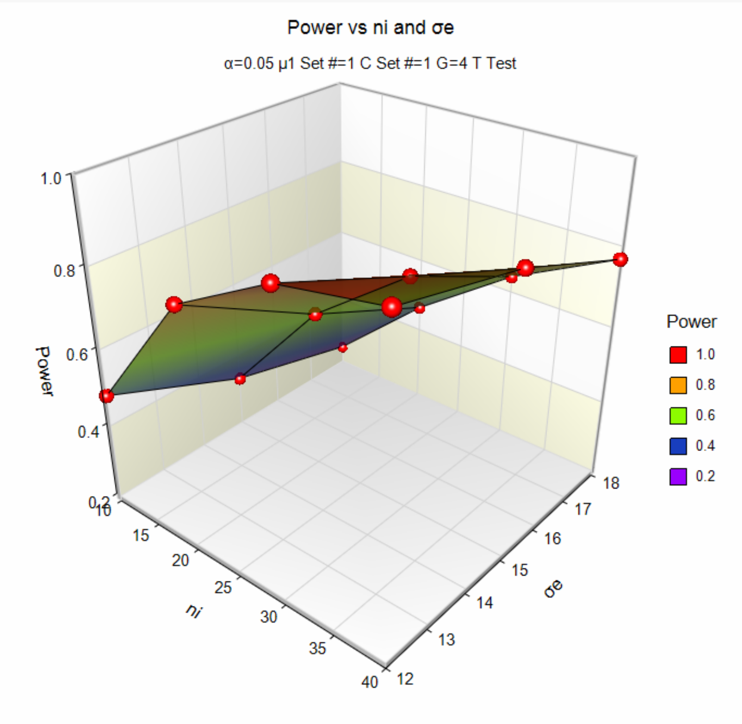
Shieh, Gwowen. 2017. Power and Sample Size Calculations for Contrast Analysis in ANCOVA. Multivariate Behavioral Research, 52:1,1-11,DOI:10.1080/00273171.2016.1219841
Kirk, Roger E. 2013. Experimental Design: Procedures for the Behavioral Sciences, 4th Edition. Sage. Washington, D.C.
Fleiss, Joseph L. 1986. The Design and Analysis of Clinical Experiments. John Wiley & Sons. New York.

This report shows the numeric results of this power study.

Plots Section

Plots





These plots give a visual presentation to the results in the Numeric Report. We can quickly see the impact on the power of increasing the sample size and the increase in the standard deviation of residuals.

Example 2 – Validation using Shieh (2017)

In Table 6 on page 7, Shieh (2017) presents several ANCOVA contrast examples in which the number of groups is 3 and the significance level is 0.05. We will use the fourth row (p = 4) as a validation run. In Table 6, Shieh sets a parameter δ (effect size) equal to 1. Using trial and error, we found that using means of 0.5, 0.5, 1.5 and contrast coefficients of 0.5, 0.5, -1 also resulted in an effect size of 1.0 when the standard deviation of the residuals is set to 1 and the number of covariates is 4. Table 6, row 4 gives a power of 0.8027 with a total sample size of 42.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For .... **Power**
Alpha.. **0.05**
G (Number of Groups) **3**
Group Allocation Input Type .... **Equal to ni (Sample Size per Group)**
ni (Sample Size Per Group)........................... **14**
μ1 Input Type.. **Enter μ1 (Group Means|H1)**
μ1 (Group Means|H1) **0.5 0.5 1.5**
Contrast Coefficients Input Type................... **List of Contrast Coefficients**
Contrast Coefficients..................................... **0.5 0.5 -1**
Standard Deviation Input Type .... **Enter σe (SD of Residuals)**
σe (SD of Residuals) **1**
Number of Covariates **4**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Solve For: Power
Number of Groups: 3
Number of Covariates: 4

Contrast Coefficients C	Power	Sample Size Total N	Sample Size Group ni	Group Means|H1 Means μ1	Group Means|H1 Contrast δ	Standard Deviation of Residuals σe	Alpha
C(1)	0.80273	42	14	μ1(1)	-1	1	0.05

The number of numerical integration intervals used in calculations was 2000.

Item	Values
C(1)	0.5, 0.5, -1
μ1(1)	0.5, 0.5, 1.5

Group Sample Size Details

n	N	Group Sample Sizes	Group Allocation Proportions
n(1)	42	14, 14, 14	0.33333, 0.33333, 0.33333

PASS has also calculated the power to be 0.8027. Thus, the procedure is validated.