Chapter 250

Chi-Square Tests

Introduction

The *Chi-square test* is often used to test whether sets of frequencies or proportions follow certain patterns. The two most common instances are tests of goodness of fit using multinomial tables and tests of independence in contingency tables.

The *Chi-square goodness of fit* test is used to test whether the distribution of a set of data follows a particular pattern. For example, the goodness-of-fit Chi-square may be used to test whether a set of values follow the normal distribution or whether the proportions of Democrats, Republicans, and other parties are equal to a certain set of values, say 0.4, 0.4, and 0.2.

The *Chi-square test for independence* in a contingency table is the most common Chi-square test. Here individuals (people, animals, or things) are classified by two (nominal or ordinal) classification variables into a two-way, contingency table. This table contains the counts of the number of individuals in each combination of the row categories and column categories. The Chi-square test determines if there is dependence (association) between the two classification variables. Hence, many surveys are analyzed with Chi-square tests.

The following table is an example of data arranged in a two-way contingency table. The rows of the table represent the stated political party of a respondent. The columns represent the respondent's answer to a question about whether they favor a certain proposition. The body of the table represents the number of individuals that fall into each cell (category). Note that the opinions of 311 individuals are recorded in this table.

(Count)	Favor Proposition A	
Political Party	**Yes**	**No**
Democrats	86	21
Republican	54	59
Others	34	57

The table below presents the row percentages for each category.

(Row Percentage)	Favor Proposition A	
Political Party	**Yes**	**No**
Democrats	80.4	19.6
Republican	47.8	52.2
Others	37.4	62.6

The Chi-square statistic tests whether the percentage of *Yes* responses remains constant across the three political parties. Notice that 80% of the Democrats said *Yes*, while only 37% of those in the Other category chose *Yes*. The Chi-square value for the above table is 41.71, which is statistically significant. Obviously, there is quite a shift in response pattern on this item across political parties.

Effect Size

We begin by defining what we will call the *effect size*. For each cell of a table containing m cells, suppose there are two proportions considered: one specified by a null hypothesis and the other specified by an alternative hypothesis. Often, the proportions specified by the alternative hypothesis are those occurring in the data. Define p_{0i} to be the proportion in cell i under the null hypothesis and p_{1i} to be the proportion in cell i under the alternative hypothesis. The effect size, w, is calculated using the formula

$$w = \sqrt{\sum_{i=1}^{m} \frac{(p_{1i} - p_{0i})^2}{p_{0i}}}.$$

The formula for computing the Chi-square value, χ^2, is

$$\chi^2 = \sum_{i=1}^{m} \frac{(O_i - E_i)^2}{E_i} = N \sum_{i=1}^{m} \frac{(p_{0i} - p_{1i})^2}{p_{0i}},$$

where N is the total count in all the cells. Hence, the relationship between w and χ^2 is

$$\chi^2 = Nw^2$$

Note that when you are dealing with a contingency table, the cell index, i, is often replaced by two indices, one representing columns and the other representing rows.

The effect size, w, was used by Cohen (1988) because it does not depend on the sample size. He sets a small value of w at 0.1, a medium value at 0.3, and a large value at 0.5. Although these are rather arbitrary settings, they are useful for planning purposes.

Chi-Square Effect Size Estimator

PASS provides a special module to aid in finding an appropriate value for w called the Chi-Square Effect Size Estimator. This module may be loaded by pressing the *CS* button near the W (Effect Size) box or from the menus by selecting *PASS* and then *Other*.

You will find that as values are typed into the body of the table, the value of the effect size (shown at the bottom in the box labeled Effect Size - W) is also changed. Using this utility program, you can quickly determine the impact of table configurations on the value of w.

For example, suppose the cell proportions under the null and alternative hypotheses are as follows:

Cell	1	2	3	4	
p_{0i}	0.25	0.25	0.25	0.25	(Null: Equal distribution across the four cells.)
p_{1i}	0.40	0.20	0.20	0.20	(Alternative: cell 1 has twice the probability as the rest.)

To calculate *w*, first create the differences (ignoring the signs since the differences will be squared

|Diff| 0.15   0.05   0.05   0.05

Next, square the differences

Diff^2   0.0225 0.0025 0.0025 0.0025

Divide by the null

X^2   0.09   0.01   0.01   0.01

When these are summed, the result is 0.12. Taking the square of 0.12 gives the value of *w* as 0.3464.

As an experiment, load the Chi-Square Effect Size Estimator and enter these values on the Multinomial Test window. Enter the p_{0i} values in the column labeled Data Values and the p_{1i} values in the column marked Hypothesized Proportions. Check that the value of *w* is 0.3464. Next, change the Data Values to 4,2,2,2 and the Hypothesized Proportions to 1,1,1,1. Check that the value of *w* is still the same.

Calculating the Power

The power is calculated as follows:

1. Find x_α such that $1 - \chi^2(x_\alpha|df) = \alpha$, where $\chi^2(x_\alpha|df)$ is the area to the left of *x* under a Chi-square distribution with *df* degrees of freedom.

2. Power = $1 - \chi'^2_{df,\lambda}$, where $\chi'^2_{k,\lambda}$ is the left-tail area of the noncentral Chi-square distribution with *k* degrees of freedom and non-centrality parameter λ. Note that $\lambda = Nw^2$.

Example 1 – Finding the Power for an Existing Contingency Table

This example will compute the power of the Chi-square test of independence of the data in the contingency table that was discussed at the beginning of this chapter. If you would like to follow along, load the Chi-Square Effect Size Estimator window, select the Contingency Table tab, enter *86, 54, 34* in the first column and *21, 59, 57* in the second column. The results are Chi-square = 41.708829, *DF* = 2, *N* = 311, and *W* = 0.366213.

We will compute the power when alpha = 0.01, 0.05, and 0.10. For evaluation purposes, we will compute the power when *N* = 20, 50, 100, and 200 as well as at 311.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For ....**Power**
DF (Degrees of Freedom).............................**2**
Alpha...**0.01 0.05 0.10**
N (Sample Size)...**20 50 100 200 311**
W (Effect Size)..**0.366213**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Reports

Numeric Results

Solve For: Power

Power	Sample Size N	Effect Size W	Chi-Square χ^2	Degrees of Freedom DF	Alpha
0.12127	20	0.3662	2.6822	2	0.01
0.29104	20	0.3662	2.6822	2	0.05
0.41007	20	0.3662	2.6822	2	0.10
0.39621	50	0.3662	6.7056	2	0.01
0.63538	50	0.3662	6.7056	2	0.05
0.74622	50	0.3662	6.7056	2	0.10
0.78214	100	0.3662	13.4112	2	0.01
0.91678	100	0.3662	13.4112	2	0.05
0.95512	100	0.3662	13.4112	2	0.10
0.98840	200	0.3662	26.8224	2	0.01
0.99795	200	0.3662	26.8224	2	0.05
0.99927	200	0.3662	26.8224	2	0.10
0.99980	311	0.3662	41.7088	2	0.01
0.99998	311	0.3662	41.7088	2	0.05
1.00000	311	0.3662	41.7088	2	0.10

Power   The probability of rejecting a false null hypothesis when the alternative hypothesis is true.
N   The size of the sample drawn from the population.
W   The effect size--a measure of the magnitude of the Chi-Square that will be detected.
χ^2 The Chi-Square value that will be detected. Note that $\chi^2 = N(W^2)$.
DF   The degrees of freedom of the corresponding Chi-Square distribution.
Alpha   The probability of rejecting a true null hypothesis.

Summary Statements

A two-way classification design (or distribution goodness of fit design) will be used to test the independence of two classification variables (or whether the table proportions are consistent with the expected values of a given distribution). The comparison will be made using a Chi-Square test. To detect an effect size of 0.3662 with a sample size of 20, 2 degrees of freedom, and a Type I error rate (α) of 0.01, the power is 0.12127.

Dropout-Inflated Sample Size

Dropout Rate	Sample Size N	Dropout-Inflated Enrollment Sample Size N'	Expected Number of Dropouts D
20%	20	25	5
20%	50	63	13
20%	100	125	25
20%	200	250	50
20%	311	389	78

Dropout Rate   The percentage of subjects (or items) that are expected to be lost at random during the course of the study and for whom no response data will be collected (i.e., will be treated as "missing"). Abbreviated as DR.

N   The evaluable sample size at which power is computed (as entered by the user). If N subjects are evaluated out of the N' subjects that are enrolled in the study, the design will achieve the stated power.

N'   The total number of subjects that should be enrolled in the study in order to obtain N evaluable subjects, based on the assumed dropout rate. N' is calculated by inflating N using the formula N' = N / (1 - DR), with N' always rounded up. (See Julious, S.A. (2010) pages 52-53, or Chow, S.C., Shao, J., Wang, H., and Lokhnygina, Y. (2018) pages 32-33.)

D   The expected number of dropouts. D = N' - N.

Dropout Summary Statements

Anticipating a 20% dropout rate, 25 subjects should be enrolled to obtain a final sample size of 20 subjects.

References

Cohen, Jacob. 1988. Statistical Power Analysis for the Behavioral Sciences, Lawrence Erlbaum Associates, Hillsdale, New Jersey.

This report shows the values of each of the parameters, one scenario per row. The definitions of each column are given in the Report Definitions section.
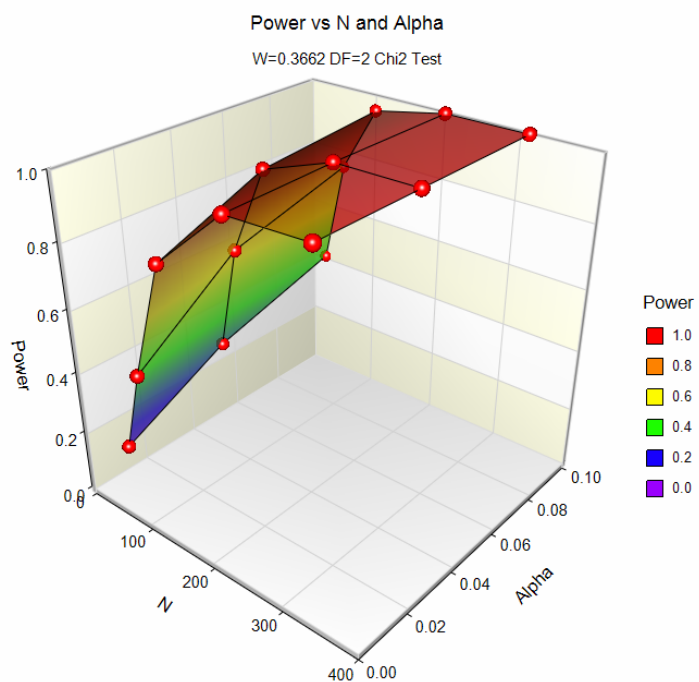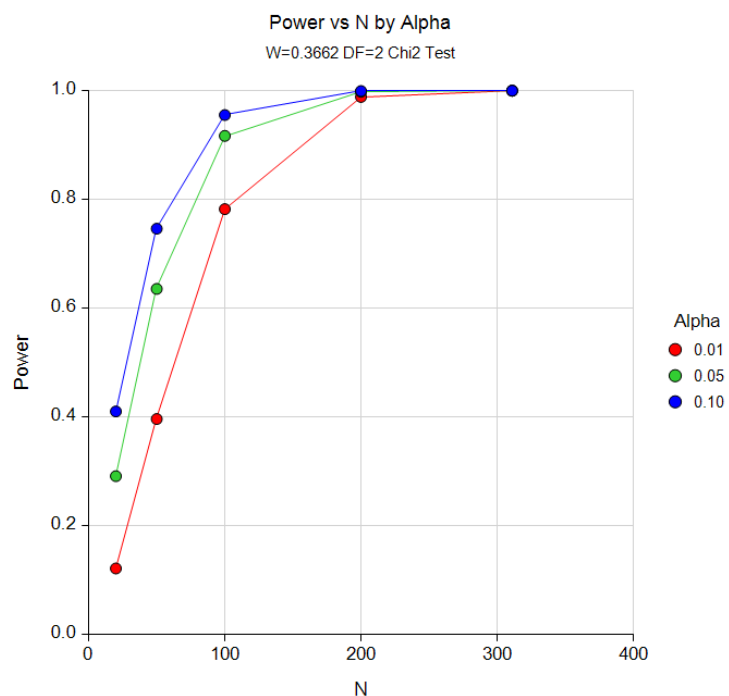
Note that in this particular example, a reasonable power of about 0.80 is reached for all values of alpha once the sample size is greater than 100.

The values from this table are displayed in the plots below.

Plots Section

Plots
_____

These plots show the relationship between sample size, power, and alpha.

Example 2 – Finding the Sample Size

A survey is being planned that will contain several questions with three possible answers: agree, neutral, disagree. The researchers are planning to analyze the questionnaires using Chi-square tests of independence in two-way contingency tables. How many respondents are needed to detect small (w = 0.1), medium (w = 0.3), or large (w = 0.5) effects if all hypothesis testing will be done at the 0.05 significance level?

Since the researchers are planning for 3-by-3 tables, DF = (3 - 1)(3 - 1) = 4.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For ....**Sample Size**
DF (Degrees of Freedom).............................**4**
Power..**0.80 0.90**
Alpha...**0.05**
W (Effect Size)...**0.1 0.3 0.5**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Solve For: Sample Size

Power	Sample Size N	Effect Size W	Chi-Square χ^2	Degrees of Freedom DF	Alpha
0.80018	1194	0.1	11.94	4	0.05
0.90010	1541	0.1	15.41	4	0.05
0.80130	133	0.3	11.97	4	0.05
0.90157	172	0.3	15.48	4	0.05
0.80243	48	0.5	12.00	4	0.05
0.90198	62	0.5	15.50	4	0.05

This report shows that for 80% power, 1194 (about 1200) respondents are needed to detect small effects, 133 respondents are needed to detect medium effects, and 48 respondents are needed to detect large effects.

Example 3 – Validation using Cohen (1988)

Cohen (1988) page 251 presents an example in which W = 0.30 and 0.40, N = 140, alpha = 0.01, and DF = 2. He gives the power as 0.75 for W = 0.3 and 0.97 for W = 0.4.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 3** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For ....**Power**
DF (Degrees of Freedom)..............................**2**
Alpha...**0.01**
N (Sample Size)...**140**
W (Effect Size)..**0.3 0.4**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results
_____

Solve For: Power
_____

Power	Sample Size N	Effect Size W	Chi-Square χ^2	Degrees of Freedom DF	Alpha
0.74841	140	0.3	12.6	2	0.01
0.96641	140	0.4	22.4	2	0.01
_____

PASS matches Cohen's power values of 0.75 and 0.97.

Example 4 – Finding the Sample Size for a Normality Goodness-of-Fit Test

A researcher is planning a study to determine if the distribution of scores on a certain test is normal. He plans to divide the test scores from his sample into five intervals of equal probability under the normal distribution using the sample mean and sample variance. After experimenting with the Chi-Square Effect Size Estimator, he decides that he must be able to detect a departure from normality of $w = 0.20$. He sets his significance level at 0.10 so that he will be lenient in his rejection of normality. He decides to focus on a power of 0.80. How large of a sample size will the researcher need?

The value of DF = 5 - 2 - 1 = 2, since there are five intervals and two parameters, mean and variance, are used.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 4** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab
_____

Solve For ....**Sample Size**
DF (Degrees of Freedom)..............................**2**
Power..**0.80**
Alpha...**0.10**
W (Effect Size)..**0.20**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results
_____

Solve For: Sample Size
_____

Power	Sample Size N	Effect Size W	Chi-Square χ^2	Degrees of Freedom DF	Alpha
0.80046	193	0.2	7.72	2	0.1
_____

This report shows that for 80% power, 193 observations are needed.