

## Chapter 408

# Confidence Intervals for Exponential Reliability

## Introduction

This routine calculates the number of events needed to obtain a specified width of a confidence interval for the reliability (proportion surviving up to time  $t$ ) when the survival time follows an exponential distribution. The calculations assume Type-II censoring, that is, the experiment is run until a set number of events occur.

## Technical Details

This procedure is based on the results of Nelson (1982) and Mathews (2010). The exponential lifetime model is based on the exponential density function

$$f(t) = \frac{1}{\theta} \exp(-t/\theta), \quad t \geq 0$$

where  $\theta$  is the mean lifetime, mean failure time, mean time to failure, or mean time between failures. This model is also parameterized in terms of failure rate,  $\lambda$  which is equal to  $1/\theta$ . In this case, the density is

$$f(t) = \lambda \exp(-\lambda t), \quad t \geq 0.$$

The cumulative exponential (failure) distribution is

$$F(t) = 1 - \exp(-t/\theta), \quad t \geq 0.$$

The survival or **reliability** function is

$$R(t) = 1 - F(t)$$

which in the case of the exponential distribution is simply

$$R(t) = \exp(-t/\theta)$$

With the assumption of Type II censoring, the maximum-likelihood estimate of  $\theta$  based on observing  $E$  failures in  $N$  items tested is

$$\bar{t} = \frac{1}{E} \sum_{k=1}^N t_k$$

where  $t_k$  is the amount of time that the  $k^{\text{th}}$  item was under test, whether the event of interest was observed in that item or not.

## Confidence Intervals for Exponential Reliability

An exact  $100(1 - \alpha)$  % confidence interval for the  $R(t)$  is given by

$$P\{LCL \leq R(t) \leq UCL\} = 1 - \alpha$$

where

$$LCL = \exp\left(-t \frac{\chi_{1-\frac{\alpha}{2}}^2}{2E\bar{t}}\right)$$

$$UCL = \exp\left(-t \frac{\chi_{\frac{\alpha}{2}}^2}{2E\bar{t}}\right)$$

Here,  $\chi_{\varphi}^2$  is the value of the chi-square random variate with  $2E$  degrees of freedom that has probability  $\varphi$  to the left.

One-sided limits may be obtained by replacing  $\alpha/2$  by  $\alpha$ .

## Confidence Interval Width

The confidence interval width, confidence level, and number of events are related in the equation

$$Width = UCL - LCL.$$

This equation can be used to find  $E$ ,  $\alpha$ , or the width.

## Confidence Level

The confidence level,  $1 - \alpha$ , has the following interpretation. If thousands of samples of  $n$  items are drawn from a population using simple random sampling and a confidence interval is calculated for each sample, the proportion of those intervals that will include the true population parameter is  $1 - \alpha$ .

## Example 1 – Calculating Sample Size

Suppose a study is planned in which the researcher wishes to construct a two-sided 95% confidence interval for  $R(t)$  such that the width of the interval is 0.05 when  $t$  is 1, 2, or 3. The percent censored is anticipated to be about 20% and the mean lifetime is 1.

### Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

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Solve For ..... **Number of Events**  
 Interval Type ..... **Two-Sided**  
 Confidence Level (1 - Alpha) ..... **0.95**  
 Percent Censored ..... **20**  
 Confidence Interval Width ..... **0.1**  
 $\theta$  (Mean Lifetime) ..... **1**  
 t (Lifetimes) ..... **1 2 3**

### Output

Click the Calculate button to perform the calculations and generate the following output.

### Numeric Reports

**Numeric Results**

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Solve For: [Number of Events](#)  
 Interval Type: Two-Sided

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Confidence Level	Number of Events E	Expected Sample Size N	Confidence Interval Width		Mean Lifetime $\theta$	Time t	Reliability at Time t R(t)	Confidence Interval Limits	
			Target	Actual				Lower	Upper
0.95	208	260	0.1	0.09980	1	1	0.36788	0.31969	0.41949
0.95	114	143	0.1	0.09987	1	2	0.13534	0.09223	0.19210
0.95	40	50	0.1	0.09893	1	3	0.04979	0.01834	0.11727

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Confidence Level      The proportion of confidence intervals (constructed with this same confidence level) that would contain the true value of  $R(t)$ .

E                              The number of events that must occur before the experiment can be stopped.

N                              Expected Sample Size. The anticipated number of subjects that must be sampled so that the desired number of events occur.

Confidence Interval Width      The distance between the lower and upper confidence interval limits.

Target Width                      The width that was requested.

Actual Width                      The calculated width. This is slightly different from the Target Width because E is an integer.

$\theta$                               Mean Lifetime. The total lifetime of all subjects divided by E, the number of events.

t                                  The time at which the reliability is calculated.

R(t)                              Reliability. The probability of surviving to time t.

Confidence Interval Limits      The confidence interval of  $R(t)$ .

## Confidence Intervals for Exponential Reliability

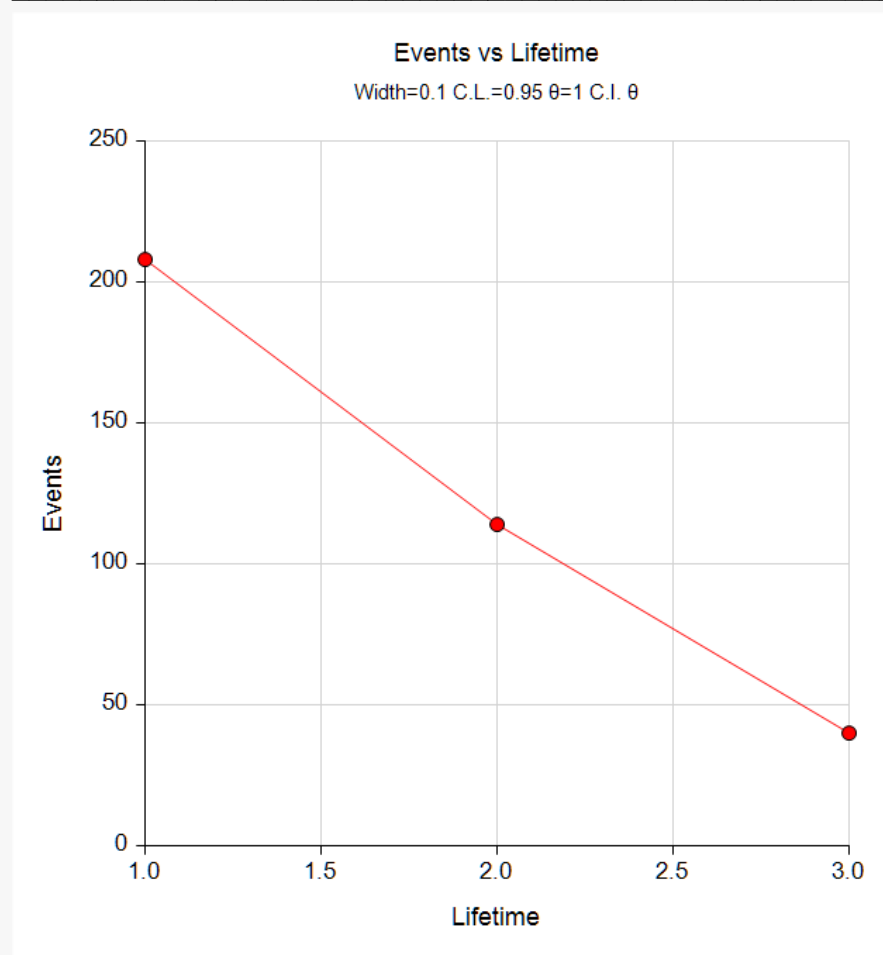
**Summary Statements**

A single-group design will be used to obtain a two-sided 95% confidence interval for the (Exponential) reliability (the probability of surviving to time 1). The Nelson (1982) and Mathews (2010) methods will be used in the confidence interval calculations. The percent censored is anticipated to be 20%. The estimated mean lifetime is assumed to be 1. To produce a confidence interval with a width of no more than 0.1, 208 events will be needed. With 20% Type-II censoring, the sample size corresponding to 208 events is 260, and it is assumed that the experiment is run until 208 events occur. The corresponding estimated reliability at time 1 is 0.36788.

**References**

Nelson, Wayne. 1982. Applied Life Data Analysis. John Wiley & Sons. New York, New York.  
Mathews, Paul. 2010. Sample Size Calculations: Practical Methods for Engineers and Scientists. Mathews Malnar and Bailey, Inc.

This report shows the calculated sample size for each of the scenarios.

**Plots Section****Plots**

This plot shows the number of events for various time points.

## Example 2 – Validation using Nelson (1982)

Nelson (1982), page 255, gives an example of that we will use to validate this procedure. In this example, the value of  $\theta$  is 6097.3, the time is 60, the confidence level is 90%, and the number of events is 10. The resulting width is 0.986. Note that Nelson rounded his calculations and obtained 0.985. The reliability is found on the same page to be 0.990. Thus, the distance is  $0.990 - 0.986 = 0.004$ .

### Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

#### Design Tab

Solve For ..... **CI Width or Distance from R(t) to Limit**  
 Interval Type ..... **Lower One-Sided**  
 Confidence Level (1 - Alpha) ..... **0.9**  
 E (Number of Events) ..... **10**  
 Percent Censored ..... **0**  
 $\theta$  (Mean Lifetime) ..... **6097.3**  
 t (Lifetimes) ..... **60**

### Output

Click the Calculate button to perform the calculations and generate the following output.

#### Numeric Results

Solve For: **CI Width or Distance from R(t) to Limit**  
 Interval Type: **Lower One-Sided**

Confidence Level	Number of Events E	Expected Sample Size N	Distance from R(t) to Lower Limit	Mean Lifetime $\theta$	Time t	Reliability at Time t R(t)	Lower Confidence Limit
0.9	10	10	0.00409	6097.3	60	0.99021	0.98612

**PASS** also calculates the distance to be 0.004.