Chapter 581

Confidence Intervals for a Percentile of a Normal Distribution using Expected Width

Introduction

This procedure calculates the sample size necessary to achieve an exact, equal-error-per-tail confidence interval width for a percentile (often called the reference limit) at a stated confidence level when the underlying data distribution is normal. It bases the optimal sample size on the assurance probability.

Technical Details

The methods used in this procedure come from Shieh (2018). This article provides useful insights into why one should use an alternative to the classic Bland-Altman interval.

Confidence Interval

Assume that you are planning a study of a population whose distribution is reasonably close to $N(\mu, \sigma^2)$ to create a confidence interval about the 100*p*th percentile of this population.

Let *X* represent the variable of interest. Suppose that a random sample of *N* observations will be obtained and the usual mean and standard deviation $(\bar{X}, S)$ will be used to estimate the 100*p*th percentile of *X* which is denoted by θ, where

$$\theta = \mu + z_p \sigma$$

Here, z_p is the 100*p*th percentile of the standard normal distribution $N(0,1)$.

An exact two-sided, 100(1 – α)% confidence interval is calculated using

$$\left(\bar{X} + t'_{\frac{\alpha}{2},[N-1,(\sqrt{N}Z_p)]}\frac{S}{\sqrt{N}}, \bar{X} + t'_{1-\frac{\alpha}{2},[N-1,(\sqrt{N}Z_p)]}\frac{S}{\sqrt{N}}\right)$$

with one-sided limits calculated by replacing $\frac{\alpha}{2}$ with α. t' is the non-central *t* distribution, with corresponding degrees of freedom and noncentrality parameter.

The width of the two-sided confidence interval is calculated as the difference between the upper and lower limits, or

$$W = \left(t'_{1-\frac{\alpha}{2},[N-1,(\sqrt{N}Z_p)]} \frac{S}{\sqrt{N}} - t'_{\frac{\alpha}{2},[N-1,(\sqrt{N}Z_p)]} \frac{S}{\sqrt{N}} \right)$$

Sample Size Based on a Bounded Expected Width

The sample size determination method used by this procedure is to determine an N that guarantees that the expected width of the confidence interval is less than a boundary value, δ. That is, we select N so that $\mathrm{E}(W) \leq \delta$. This leads to the following expression from which the desired N can be determined by a simple binary search.

$$\frac{t'_{1-\frac{\alpha}{2},[N-1,(\sqrt{N}Z_p)]} - t'_{\frac{\alpha}{2},[N-1,(\sqrt{N}Z_p)]}}{c\sqrt{N}} \leq \frac{\delta}{\sigma}$$

where

$$c = \frac{\Gamma\left(\frac{N-1}{2}\right)\sqrt{\frac{N-1}{2}}}{\Gamma\left(\frac{N}{2}\right)}$$

Example 1 – Calculating Sample Size

Suppose a study is planned in which the researcher wishes to construct an exact, two-sided 95% confidence interval for the 95th percentile such that the interval width is guaranteed to be no wider than 1, 2, or 3 units. The standard deviation is between 5 and 7.

The goal is to determine the necessary sample size.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the procedure window. You may then make the appropriate entries as listed below, or open **Example 1** by going to the **File** menu and choosing **Open Example Template**.

Option	Value
Design Tab	
Solve For	**Sample Size (N)**
Confidence Level (1 - α)	**0.95**
δ (Upper Bound of Expected C.I. Width)	**1 2 3**
p (Percentile Proportion)	**0.9**
σ (Standard Deviation)	**5 6 7**

Annotated Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Numeric Results

Solve For: Sample Size (N)
E(W) Requirement: E(W) ≤ δ
Interval Type: Two-Sided Confidence Interval

Confidence Level 1 - α	Sample Size N	Percentile Proportion p	Upper Bound of Expected C.I. Width Target δT	Upper Bound of Expected C.I. Width Actual δA	Standard Deviation σ
0.95	703	0.9	1	1.000	5
0.95	1011	0.9	1	1.000	6
0.95	1375	0.9	1	1.000	7
0.95	178	0.9	2	1.998	5
0.95	255	0.9	2	1.999	6
0.95	346	0.9	2	1.999	7
0.95	81	0.9	3	2.991	5
0.95	115	0.9	3	2.997	6
0.95	156	0.9	3	2.992	7

Report Definitions

1 - α is the confidence level of the confidence interval of the percentile.
N is the sample size of the study.
p is the percentile proportion. It is the proportion of observations that fall at or below the 100pth percentile value. For example, a value of 0.7 indicates the 70th percentile.
δT is the target upper bound of the expected width of the confidence interval.
δA is the actual upper bound of the expected width of the confidence interval. It may be different from the target value because of the discrete nature of N.
σ is the standard deviation of the population about which the percentile is being constructed.

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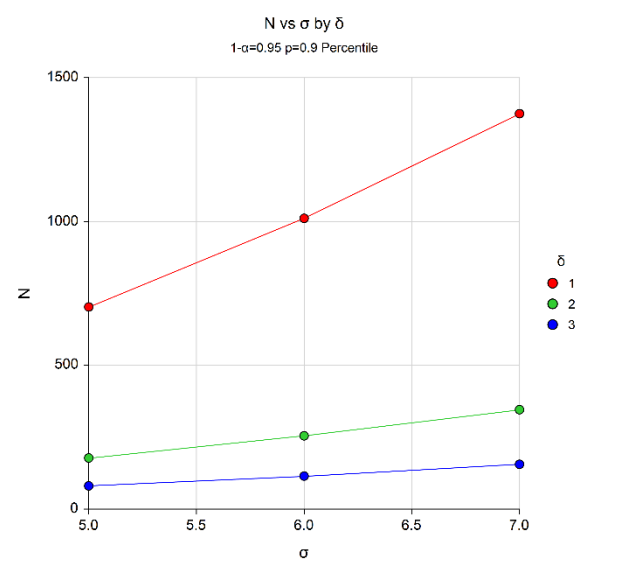
References
Shieh, Gwowen. 2018. 'The appropriateness of Bland-Altman's approximate confidence intervals for limits of agreement.' BMC Medical Research Methodology. 18,45,1. https://doi.org/10.1186/s12874-018-0505-y
Hahn, G. J. and Meeker, W.Q. 1991. Statistical Intervals. John Wiley & Sons. New York.

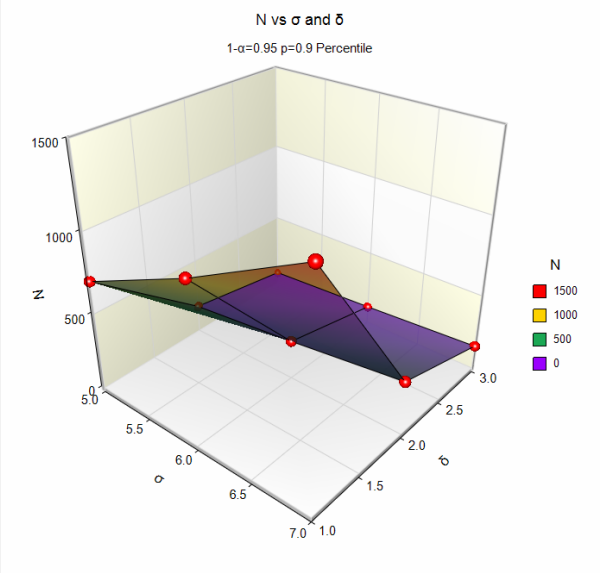
Summary Statements
A sample size of 703 produces a two-sided 95% confidence interval of the 90th percentile with an expected width of no more than 1. The standard deviation is 5.

This report shows the calculated sample size for each of the scenarios.

Chart Section

Chart Section





These plots show the sample sizes required for the various scenarios.

Example 2 – Validation using Shieh (2018)

Shieh (2018) page 7 gives an example of an exact interval calculation for a confidence interval for the 97.5th percentile when the confidence coefficient is 95%, the standard deviation is 19.61, and the expected confidence interval width lower bound is 9.805. The resulting sample size is 183.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the procedure window. You may then make the appropriate entries as listed below, or open **Example 2** by going to the **File** menu and choosing **Open Example Template**.

Option	Value
Design Tab	
Solve For	**Sample Size (N)**
Confidence Level (1 - α)	**0.95**
δ (Upper Bound of Expected C.I. Width)	**9.805**
p (Percentile Proportion)	**0.975**
σ (Standard Deviation)	**19.61**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Numeric Results

Solve For: Sample Size (N)
E(W) Requirement: E(W) ≤ δ
Interval Type: Two-Sided Confidence Interval

Confidence Level 1 - α	Sample Size N	Percentile Proportion p	Upper Bound of Expected C.I. Width Target δT	Upper Bound of Expected C.I. Width Actual δA	Standard Deviation σ
0.95	183	0.975	9.805	9.799	19.61

The sample size computed by **PASS** is also 183. This validates the procedure.