Chapter 409

Confidence Intervals for the Exponential Hazard Rate

Introduction

This routine calculates the number of events needed to obtain a specified width of a confidence interval for the hazard rate of an exponential distribution at a given level of confidence. The calculations assume Type-II censoring, that is, the experiment is run until a set number of events occur.

Technical Details

This procedure is based on the results of Nelson (1982) and Lawless (2003). The exponential lifetime model is based on the exponential density function

$$f(t) = \lambda \exp(-\lambda t), \ \ t \geq 0$$

where λ is the hazard (failure) rate.

The cumulative exponential distribution is

$$F(t) = 1 - \exp(-\lambda t), \ \ t \geq 0.$$

The survival or reliability function is

$$R(t) = 1 - F(t)$$

which in the case of the exponential distribution results in

$$R(t) = \exp(-\lambda t)$$

With the assumption of Type II censoring, the maximum-likelihood estimate of λ based on observing E failures in N items tested is

$$h = \frac{E}{\sum_{k=1}^{N} t_k}$$

where t_k is the amount of time that the kth subject was under test, whether the event of interest was observed in that subject or not.

An exact 100(1 – α) % confidence interval for λ is given by

$$P\left(\frac{h\chi^2_{\alpha/2}}{2E} \leq \lambda \leq \frac{h\chi^2_{1-\alpha/2}}{2E}\right) = 1 - \alpha$$

where χ^2_φ is the value of the chi-square random variate with *2E* degrees of freedom that has probability φ to the left.

One-sided limits may be obtained by replacing $\alpha/2$ by α.

Confidence Interval Width

The confidence interval width, confidence level, and number of events are related in the equation

$$Width = UCL - LCL$$

where LCL and UCL are the lower and upper confidence limits.

This equation can be used to find *E*, α, or the width.

Confidence Level

The confidence level, 1 – α, has the following interpretation. If thousands of samples of *n* items are drawn from a population using simple random sampling and a confidence interval is calculated for each sample, the proportion of those intervals that will include the true population parameter is 1 – α.

Example 1 – Calculating Sample Size

Suppose a study is planned in which the researcher wishes to construct a two-sided 95% confidence interval for the hazard rate such that the width of the interval is 0.4 or 0.6. The hazard rate is estimated to be between 1 and 3. The percent censored is anticipated to be about 20%.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For ....**Number of Events**
Interval Type ....**Two-Sided**
Confidence Level (1 - Alpha) **0.95**
Percent Censored...**20**
Confidence Interval Width.............................**0.4 0.6**
λ (Hazard Rate) **1 1.5 2 2.5 3**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Reports

Numeric Results

Solve For: Number of Events
Interval Type: Two-Sided

Confidence Level	Number of Events E	Expected Sample Size N	Confidence Interval Width		Hazard Rate λ	Confidence Interval Limits	
			Target	Actual		Lower	Upper
0.95	96	120	0.4	0.39971	1.0	0.81000	1.20971
0.95	216	270	0.4	0.39991	1.5	1.30662	1.70653
0.95	384	480	0.4	0.39998	2.0	1.80494	2.20492
0.95	601	752	0.4	0.39968	2.5	2.30410	2.70378
0.95	865	1082	0.4	0.39980	3.0	2.80338	3.20319
0.95	43	54	0.6	0.59657	1.0	0.72370	1.32027
0.95	96	120	0.6	0.59957	1.5	1.21500	1.81457
0.95	171	214	0.6	0.59922	2.0	1.71146	2.31068
0.95	267	334	0.6	0.59954	2.5	2.20909	2.80864
0.95	384	480	0.6	0.59998	3.0	2.70741	3.30739

Confidence Level   The proportion of confidence intervals (constructed with this same confidence level) that would contain the true value of λ.
E   The number of events that must occur before the experiment can be stopped.
N   Expected Sample Size. The anticipated number of subjects that must be sampled so that the desired number of events occur.

Confidence Interval Width	The distance between the lower and upper confidence interval limits.
Target Width	The width that was requested.
Actual Width	The calculated width. This is slightly different from the Target Width because E is an integer.
λ	Hazard Rate. The number of events E divided by the total lifetime of all subjects T.
Confidence Interval Limits	The confidence interval of λ.

Summary Statements
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A single-group design will be used to obtain a two-sided 95% confidence interval for the hazard rate of an Exponential distribution. The Nelson (1982) and Lawless (2003) methods will be used in the confidence interval calculations. The percent censored is anticipated to be 20%. The estimated hazard rate is assumed to be 1. To produce a confidence interval with a width of no more than 0.4, 96 events will be needed. With 20% Type-II censoring, the sample size corresponding to 96 events is 120, and it is assumed that the experiment is run until 96 events occur.

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References
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Nelson, Wayne. 1982. Applied Life Data Analysis. John Wiley & Sons. New York, New York.
Lawless, Jerald F. 2003. Statistical Models and Methods for Lifetime Data, 2nd Edition. John Wiley, New York.
Mathews, Paul. 2010. Sample Size Calculations: Practical Methods for Engineers and Scientists. Mathews Malnar and Bailey, Inc.
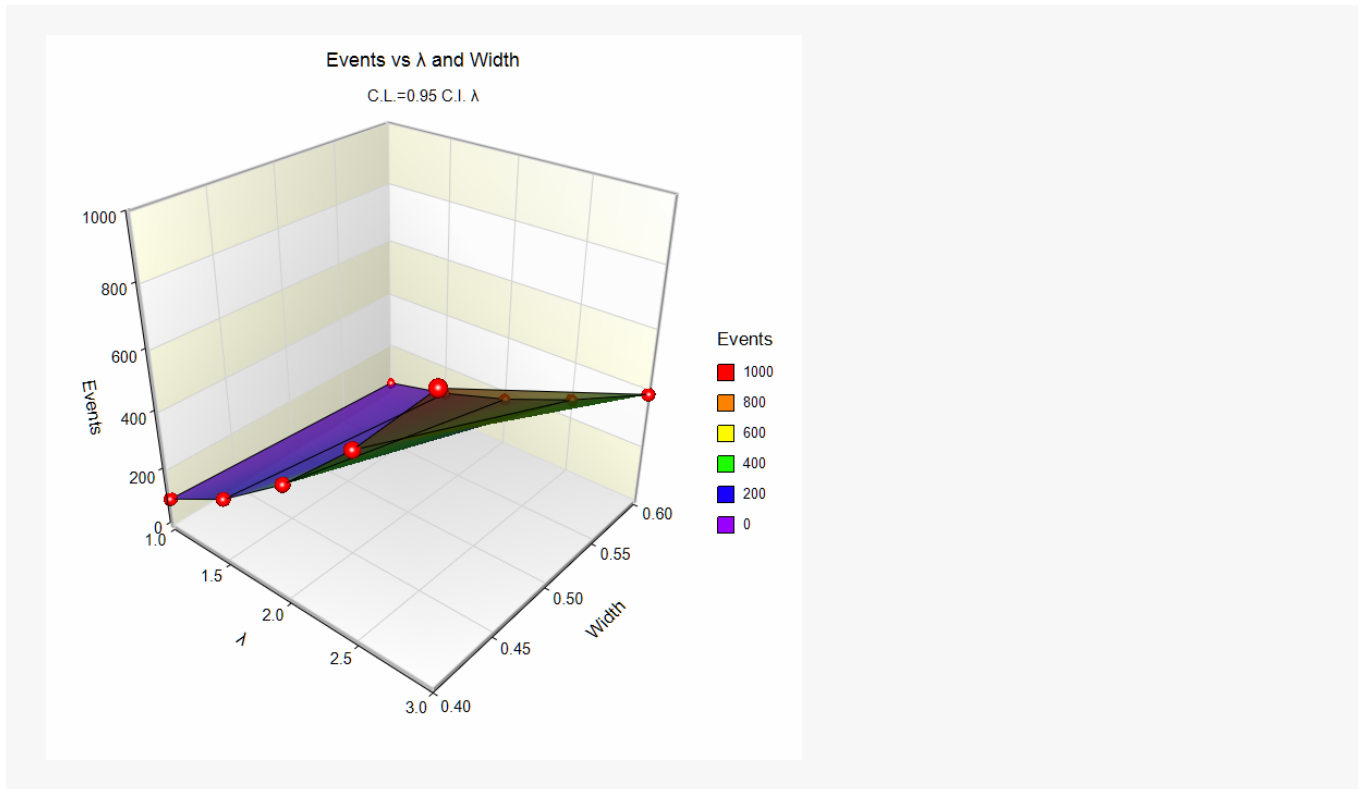───

This report shows the calculated number of events for each of the scenarios.

Plots Section

Plots
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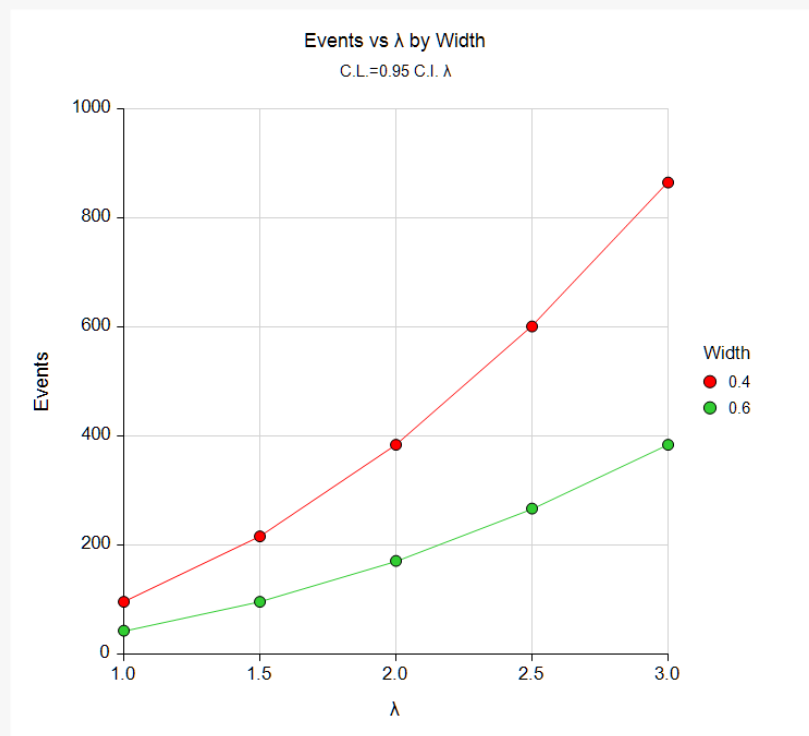
Confidence Intervals for the Exponential Hazard Rate

These plots show the number of events for various hazard rates.

Example 2 – Validation using Nelson (1982)

Nelson (1982), page 254, gives an example of a calculation of an upper one-sided confidence interval for the hazard rate. In this example the value of λ is 0.000164, the confidence level is 90%, the upper confidence limit is 0.000233, and the width is 0.000069. The number of events is 10.

Note that Nelson erroneously calculated the upper limit as 1/4092.1. Using his own numbers from the bottom of page 252, the upper limit should be 1/4292.1.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For ....**Number of Events**
Interval Type ....**Upper One-Sided**
Confidence Level (1 - Alpha) **0.9**
Percent Censored..**0**
Distance from λ to Limit **0.000069**
λ (Hazard Rate) **0.000164**

Reports Tab

Widths and Distances Decimals **6**
λ (Hazard Rate) Decimals.............................**6**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results
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Solve For: Number of Events
Interval Type: Upper One-Sided
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| Confidence Level | Number of Events E | Expected Sample Size N | Distance from λ to Upper Limit | | Hazard Rate λ | Upper Confidence Limit |
			Target	Actual		
0.9	10	10	0.000069	0.000069	0.000164	0.000233

PASS also calculates the number of events to be 10.