Chapter 406

Confidence Intervals for the Exponential Lifetime Mean

Introduction

This routine calculates the number of events needed to obtain a specified width of a confidence interval for the mean of an exponential distribution at a given level of confidence. The calculations assume Type-II censoring, that is, the experiment is run until a set number of events occur.

Technical Details

This procedure is based on the results of Mathews (2010) and Lawless (2003). The exponential lifetime model is based on the exponential density function

$$f(t) = \frac{1}{\theta}\exp(-t/\theta), \ \ t \geq 0$$

where θ is the mean lifetime, mean failure time, mean time to failure, or mean time between failures. This model is also parameterized in terms of failure rate, λ which is equal to $1/\theta$. In this case, the density is

$$f(t) = \lambda\exp(-\lambda t), \ \ t \geq 0.$$

The cumulative exponential distribution is

$$F(t) = 1 - \exp(-t/\theta), \ \ t \geq 0.$$

The survival or reliability function is

$$R(t) = 1 - F(t)$$

which in the case of the exponential distribution results in

$$R(t) = \exp(-t/\theta)$$

With the assumption of Type II censoring, the maximum-likelihood estimate of θ based on observing E failures in N items tested is

$$\bar{t} = \frac{1}{E}\sum_{k=1}^{N} t_k$$

where t_k is the amount of time that the k^{th} item was under test, whether the event of interest was observed in that item or not.

An exact 100(1 – α) % confidence interval for θ is given by

$$P\left(\frac{2E\bar{t}}{\chi^2_{1-\alpha/2}} \leq \theta \leq \frac{2E\bar{t}}{\chi^2_{\alpha/2}}\right) = 1 - \alpha$$

where χ^2_{φ} is the value of the chi-square random variate with *2E* degrees of freedom that has probability φ to the left.

One-sided limits may be obtained by replacing $\alpha/2$ by α.

Confidence Interval Width

The confidence interval width, confidence level, and number of events are related in the equation

$$Width = UCL - LCL$$

where LCL and UCL are the lower and upper confidence limits.

This equation can be used to find *E*, α, or the width.

Confidence Level

The confidence level, 1 – α, has the following interpretation. If thousands of samples of *n* items are drawn from a population using simple random sampling and a confidence interval is calculated for each sample, the proportion of those intervals that will include the true population parameter is 1 – α.

Example 1 – Calculating Sample Size

Suppose a study is planned in which the researcher wishes to construct a two-sided 95% confidence interval for θ such that the relative width of the interval is 0.05, 0.10, 0.15, and 0.20. The percent censored is anticipated to be about 20%.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For ....**Number of Events**
Interval Type ....**Two-Sided**
Confidence Level (1 - Alpha) **0.95**
Percent Censored...**20**
Confidence Interval Width.............................**0.05 to 0.2 by 0.05**
θ (Mean Lifetime)..**1**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Reports

Numeric Results

Solve For: Number of Events
Interval Type: Two-Sided

Confidence Level	Number of Events E	Expected Sample Size N	Confidence Interval Width Target	Confidence Interval Width Actual	Mean Lifetime θ	Confidence Interval Limits Lower	Confidence Interval Limits Upper
0.95	6151	7689	0.05	0.05000	1	0.97547	1.02547
0.95	1541	1927	0.10	0.09998	1	0.95189	1.05187
0.95	687	859	0.15	0.14996	1	0.92924	1.07920
0.95	388	485	0.20	0.19996	1	0.90751	1.10747

Confidence Level The proportion of confidence intervals (constructed with this same confidence level) that would contain the true value of θ.
E The number of events that must occur before the experiment can be stopped.
N Expected Sample Size. The anticipated number of subjects that must be sampled so that the desired number of events occur.
Confidence Interval Width The distance between the lower and upper confidence interval limits.
Target Width The width that was requested.
Actual Width The calculated width. This is slightly different from the Target Width because E is an integer.
θ Mean Lifetime. The total lifetime of all subjects divided by E, the number of events.
Confidence Interval Limits The confidence interval of θ.

Summary Statements

A single-group design will be used to obtain a two-sided 95% confidence interval for the (Exponential) mean lifetime. The Lawless (2003) and Mathews (2010) methods will be used in the confidence interval calculations. The percent censored is anticipated to be 20%. The estimated mean lifetime is assumed to be 1. To produce a confidence interval with a width of no more than 0.05, 6151 events will be needed. With 20% Type-II censoring, the sample size corresponding to 6151 events is 7689, and it is assumed that the experiment is run until 6151 events occur.

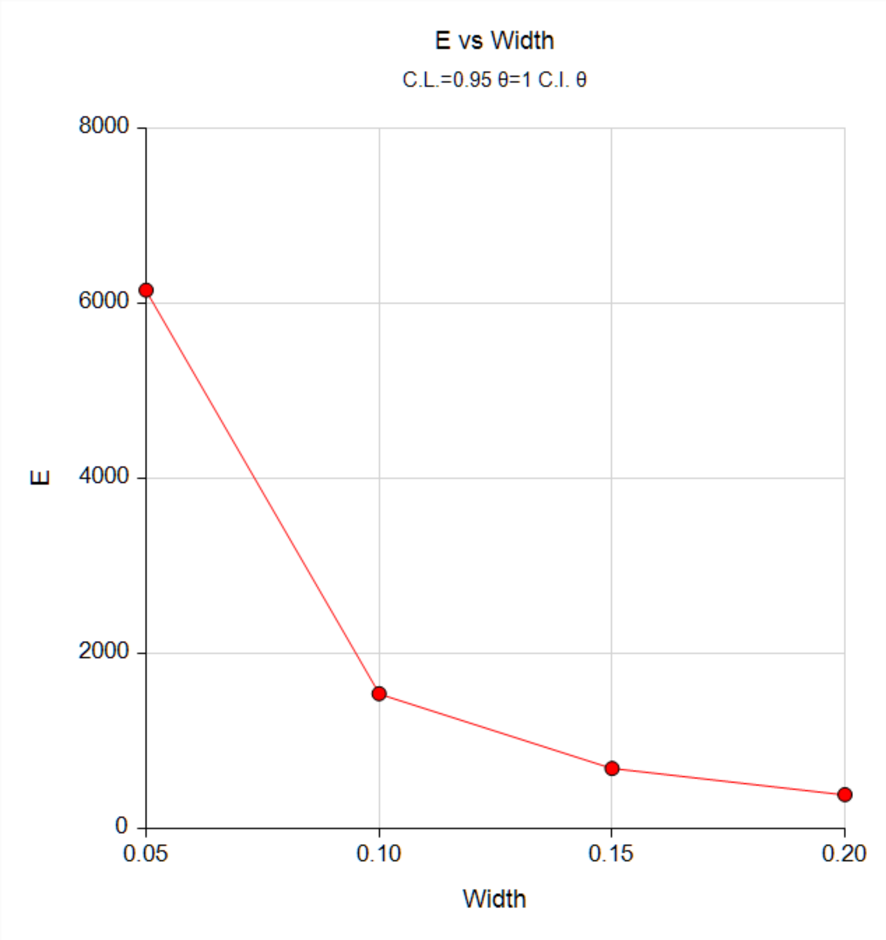
References

Lawless, Jerald F. 2003. Statistical Models and Methods for Lifetime Data, 2nd Edition. John Wiley, New York.
Mathews, Paul. 2010. Sample Size Calculations: Practical Methods for Engineers and Scientists. Mathews Malnar and Bailey, Inc.

This report shows the calculated sample size for each of the scenarios.

Plots Section

Plots



This plot shows the number of events, E, for various widths.

Example 2 – Validation using Mathews (2010)

Mathews (2010), page 192, gives an example of a sample size calculation. In this example the value of θ is 1.0, the confidence level is 95%, and the width is 0.40 (plus or minus 0.2). The resulting number of events is 97. Note that Mathews uses a normal approximation to the chi-square distribution which makes his results a little different than ours.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For .... **Number of Events**
Interval Type .... **Two-Sided**
Confidence Level (1 - Alpha) **0.95**
Percent Censored... **20**
Confidence Interval Width............................. **0.4**
θ (Mean Lifetime).. **1**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Solve For: Number of Events
Interval Type: Two-Sided

Confidence Level	Number of Events E	Expected Sample Size N	Confidence Interval Width: Target	Confidence Interval Width: Actual	Mean Lifetime θ	Confidence Interval Limits: Lower	Confidence Interval Limits: Upper
0.95	100	125	0.4	0.39937	1	0.82968	1.22904

PASS calculates the number of events to be 100. The difference between 100 and the 97 that Mathews obtained is because Mathews used a normal approximation to the chi-square distribution. **PASS** used exact chi-square results.