Chapter 566

Equivalence Tests for One-Way Analysis of Variance Assuming Equal Variances

Introduction

This procedure computes power and sample size of equivalence tests of the means of two or more groups which are analyzed using a noncentral F-test. The results in this chapter come from Shieh (2016), Jan and Shieh (2019), and Wellek (2010).

Technical Details for the One-Way ANOVA Equivalence Test

Suppose G groups each have a normal distribution and with means $\mu_1, \mu_2, \ldots, \mu_G$ and common variance σ^2. Let $n_1, n_2, \ldots, n_G$ denote the sample size of each group and let N denote the total sample size of all groups. The multigroup equivalence problem requires one to show that the means are sufficiently close to each other. Wellek (2010) accomplished this by defining a set of equivalence means, $\mu_{01}, \mu_{02}, \ldots, \mu_{0G}$, that are as far apart as possible and still be termed equivalent. He then summarized the means using their weighted variance

$$\sigma_{m0}^2 = \sum_{i=1}^{G} \left(\frac{n_i}{N}\right)(\mu_{0i} - \bar{\mu}_0)^2$$

The corresponding variance of the group means under the alternative hypothesis is given by

$$\sigma_{m1}^2 = \sum_{i=1}^{G} \left(\frac{n_i}{N}\right)(\mu_{1i} - \bar{\mu}_1)^2$$

where $\bar{\mu}_1$ is the weighted mean

$$\bar{\mu}_1 = \sum_{i=1}^{G} \left(\frac{n_i}{N}\right)\mu_{1i}$$

Given the above terminology, Wellek (2010) and Shieh (2016) suggested testing the hypothesis of mean equivalence using

$$H_0: f \geq f_0 \text{ versus } H_1: f < f_0$$

where f^2 represents the usual F-test from a one-way design and $f_0^2 = \sigma_{m0}^2/\sigma^2$ is the equivalence bound defined above.

Under H_0, the usual F statistic, denoted F^*, is assumed to follow the noncentral F distribution

$$F^* \sim F'_{G-1,N-G,\Lambda_0}$$

where $\Lambda_0 = Nf_0^2$. Note that Λ_0 does not depend on data. Rather, it depends on the user specified set of equivalence values.

The value of F^* is given by

$$F^* = \frac{SSM/(G-1)}{SSE/(N-G)}$$

where *SSM* is the sum of squares of treatment means and *SSE* is the sum of squares of error.

The null hypothesis is rejected at the significance level α if $F^* < F'_{1-\alpha,G-1,N-G,\Lambda_0}$.

The power function of this test statistic at a particular set of means is given by

$$\text{Power} = \Pr\left[F'_{G-1,N-G,\Lambda_1} < F'_{1-\alpha,G-1,N-G,\Lambda_0}\right]$$

where $\Lambda_1 = Nf_1^2$ and $f_1^2 = \sigma_{m1}^2/\sigma^2$. This can easily be computed using our noncentral *F* cumulative distribution function. You have to be careful that $\sigma_{m1}^2 < \sigma_{m0}^2$.

If a sample size is desired, it can be determined using a standard binary search algorithm.

Example 1 – Finding Power

An experiment is being designed to assess the equivalence of the means of four groups using a noncentral *F* test with a significance level of 0.05. Previous studies have shown that the standard deviation is about 2. The variation allowed in equivalent means is represented by the values {5, 5, 7, 7}. Treatment means of {5, 5, 6, 6} represent the alternative hypothesis. To better understand the relationship between power and sample size, the researcher wants to compute the power for several group sample sizes between 10 and 70. The sample sizes will be equal across all groups.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the procedure window. You may then make the appropriate entries as listed below, or open **Example 1** by going to the **File** menu and choosing **Open Example Template**.

Option	Value
Design Tab	
Solve For	**Power**
Alpha	**0.05**
G (Number of Groups)	**4**
Group Allocation Input Type	**Equal to ni (Sample Size per Group)**
ni (Sample Size per Group)	**10 20 30 40 50 60 70**
μ0 Input Type	**Enter μ0 (Group Means|H0)**
μ0 (Group Means|H0)	**5 5 7 7**
μ1 Input Type	**Enter μ1 (Group Means|H1)**
μ1 (Group Means|H1)	**5 5 6 6**
σ (Standard Deviation)	**2**

Annotated Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Numeric Results
Number of Groups 4

	Sample Size		Group Means H0		Group Means H1			SD of Standardized Means		
	Total	Grp	Means	SD(μ0)	Means	SD(μ1)	Std Dev	H0	H1	
Power	N	ni	μ0	σm0	μ1	σm1	σ	f0	f1	Alpha
0.38245	40	10	μ0(1)	1	μ1(1)	0.5	2	0.5	0.25	0.05
0.65712	80	20	μ0(1)	1	μ1(1)	0.5	2	0.5	0.25	0.05
0.81888	120	30	μ0(1)	1	μ1(1)	0.5	2	0.5	0.25	0.05
0.90803	160	40	μ0(1)	1	μ1(1)	0.5	2	0.5	0.25	0.05
0.95474	200	50	μ0(1)	1	μ1(1)	0.5	2	0.5	0.25	0.05
0.97828	240	60	μ0(1)	1	μ1(1)	0.5	2	0.5	0.25	0.05
0.98979	280	70	μ0(1)	1	μ1(1)	0.5	2	0.5	0.25	0.05

Value Lists

Name	Value
μ0(1)	5, 5, 7, 7
μ1(1)	5, 5, 6, 6

Group Sample Size Details

n	N	Group Sample Sizes	Group Allocation Proportions
n(1)	40	10, 10, 10, 10	0.25, 0.25, 0.25, 0.25
n(2)	80	20, 20, 20, 20	0.25, 0.25, 0.25, 0.25
n(3)	120	30, 30, 30, 30	0.25, 0.25, 0.25, 0.25
n(4)	160	40, 40, 40, 40	0.25, 0.25, 0.25, 0.25
n(5)	200	50, 50, 50, 50	0.25, 0.25, 0.25, 0.25
n(6)	240	60, 60, 60, 60	0.25, 0.25, 0.25, 0.25
n(7)	280	70, 70, 70, 70	0.25, 0.25, 0.25, 0.25

References

Jan, S-L and Shieh, G. 2019. 'On the Extended Welch Test for Assessing Equivalence of Standardized Means'. Statistics in Biopharmaceutical Research. DOI:10.1080/19466315.2019.1654915
Shieh, G. 2016. 'A comparative appraisal of two equivalence tests for multiple standardized effects'. Computer Methods and Programs in Biomedicine, Vol 126, Pages 110-117. http://dx.doi.org/10.1016/j.cmpb.2015.12.004
Wellek, Stefan. 2010. Testing Statistical Hypotheses of Equivalence and Noninferiority, 2nd Edition. CRC Press. New York.

Report Definitions

Power is the probability of rejecting a false null hypothesis of non-equivalence in favor of the alternative hypothesis of equivalence.
N, the Total Sample Size, is the total number of subjects in the study.
ni, the Sample Size per Group, is the number of items sampled from each group in the study.
μ0, the Group Means | H0, is the set name and number of the group means under the null hypothesis. These values are used to form the equivalence boundary ω0.
σm0, the SD of Group Means μ0 | H0, is the standard deviation of the equivalence means. Note that this value also depends on the group sample sizes.
μ1, the Group Means | H1, is the set name and number of the group means under the alternative hypothesis. This is the set of means at which the power is calculated.
σm1, the SD of Group Means μ1 | H1, is the standard deviation of the group means assumed by the alternative hypothesis. Note that this value also depends on the group sample sizes.
Std Dev σ is the common standard deviation of the responses within a group.
f0, the SD of Standardized Means | H0, is the standard deviation of the standardized means assumed by the null hypothesis, H0. This is the upper bound of equivalence. Note that you must have f0 < f1.
f1, the SD of Standardized Means | H1, is the standard deviation of the standardized means assumed by the alternative hypothesis, H1. Note that you must have f0 < f1.
Alpha is the significance level of the test: the probability of rejecting the null hypothesis of non-equivalent means when it is actually true.

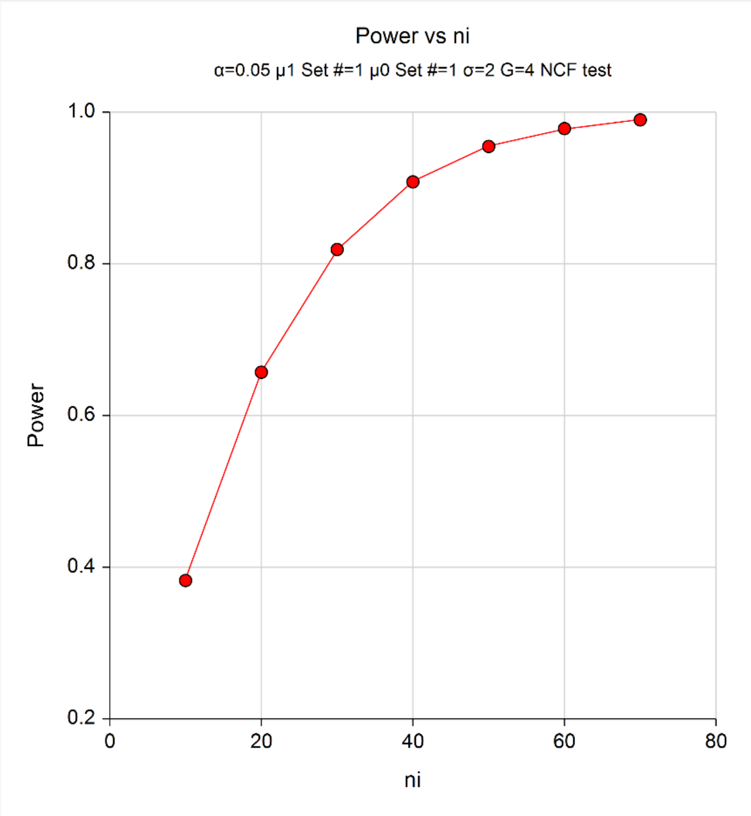
Summary Statements

In an equivalence, one-way ANOVA study, a sample of 40 subjects, divided among 4 groups, achieves a power of 38%. This power assumes a noncentral F test with a significance level of 0.05. The group subject counts are 10, 10, 10, 10. The group means under the null hypothesis are 5, 5, 7, 7. These means define the equivalence boundary. The group means under the alternative hypothesis are 5, 5, 6, 6. The standard deviation of hypothesized means under the alternative hypothesis is 0.5. The standard deviation of equivalence means is 1. The common standard deviation of the responses is 2. The standard deviation of the standardized means under the null hypothesis is 0.5. The standard deviation of the standardized means under the alternative hypothesis is 0.25.

This report shows the numeric results of this power study.

Chart Section

Chart Section



This plot gives a visual presentation of the results in the Numeric Report.

Example 2 – Finding the Sample Size Necessary to Reject

Continuing with the last example, we will determine how large the sample size would need to have been for alpha = 0.05 and power = 0.80 or 0.9.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the procedure window. You may then make the appropriate entries as listed below, or open **Example 2** by going to the **File** menu and choosing **Open Example Template**.

Option	Value
Design Tab	
Solve For	**Sample Size**
Power	**0.8 0.9**
Alpha	**0.05**
G (Number of Groups)	**4**
Group Allocation Input Type	**Equal (n1 = ··· = nG)**
μ0 Input Type	**Enter μ0 (Group Means|H0)**
μ0 (Group Means|H0)	**5 5 7 7**
μ1 Input Type	**Enter μ1 (Group Means|H1)**
μ1 (Group Means|H1)	**5 5 6 6**
σ (Standard Deviation)	**2**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Numeric Results

Number of Groups 4

	Sample Size		Group Means				Std	SD of Standardized Means		
			H0		H1					
	Total	Grp	Means	SD(μ0)	Means	SD(μ1)	Dev	H0	H1	
Power	N	ni	μ0	σm0	μ1	σm1	σ	f0	f1	Alpha
0.80657	116	29	μ0(1)	1	μ1(1)	0.5	2	0.5	0.25	0.05
0.90143	156	39	μ0(1)	1	μ1(1)	0.5	2	0.5	0.25	0.05

Value Lists

Name	Value
μ0(1)	5, 5, 7, 7
μ1(1)	5, 5, 6, 6

Group Sample Size Details

n	N	Group Sample Sizes	Group Allocation Proportions
n(1)	116	29, 29, 29, 29	0.25, 0.25, 0.25, 0.25
n(2)	156	39, 39, 39, 39	0.25, 0.25, 0.25, 0.25

The required sample size jumps from 116 to 156 as the power is increased.

Example 3 – Validation using Shieh (2016)

Shieh (2016) page 114 presents an example in which alpha = 0.05, G = 3, σ = 1, σm1 = 0.05, and σm0 = 0.25, and power = 0.6503. The resulting sample size is 48 per group for a total of 144.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the procedure window. You may then make the appropriate entries as listed below, or open **Example 3** by going to the **File** menu and choosing **Open Example Template**.

Option	Value
Design Tab	
Solve For	**Sample Size**
Power	**0.6503**
Alpha	**0.05**
G (Number of Groups)	**3**
Group Allocation Input Type	**Equal (n1 = ⋯ = nG)**
μ0 Input Type	**Enter σm0 (SD of μ0)**
σm0 (SD of μ0)	**0.25**
μ1 Input Type	**Enter σm1 (SD of μ1)**
σm1 (SD of μ1)	**0.05**
σ (Standard Deviation)	**1**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Numeric Results
Number of Groups 3

	Sample Size		Means		Std	SD of Standardized Means		
	Total	Grp	H0 SD(μ0)	H1 SD(μ1)	Dev	H0	H1	
Power	N	ni	σm0	σm1	σ	f0	f1	Alpha
0.6503	144	48	0.25	0.05	1	0.25	0.05	0.05

Group Sample Size Details

n	N	Group Sample Sizes	Group Allocation Proportions
n(1)	144	48, 48, 48	0.333, 0.333, 0.333

PASS also found N = 144. The procedure is validated.

Example 4 – Comparison with Two-Sample Equivalence Test

It is interesting to compare the results from this procedure with G = 2 with results from the *Two-Sample T-Tests for Equivalence Assuming Equal Variance* procedure. Here is a copy of an analysis run for that procedure.

Numeric Results for Two One-Sided Equal-Variance T-Tests
δ = μ1 - μ2 = μT - μR
Hypotheses: H0: δ ≤ EL or δ ≥ EU vs. H1: EL < δ < EU

Target Power	Actual Power	N1	N2	N	Lower Equiv Limit EL	Upper Equiv Limit EU	δ	σ	Alpha
0.9	0.90009	2707	2707	5414	-10	10	2	100	0.05

To duplicate this example, we set alpha = 0.05, G = 2, power = 0.9, σ = 100, $\mu_1 = 0$, $\mu_2 = 2$, $E_1 = 0$, and $E_2 = 10$. The sample size will be calculated.

Note that we set the difference of the two means to match the value of δ and the difference between the equivalence means to be one-half the difference between the two equivalence bounds.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the procedure window. You may then make the appropriate entries as listed below, or open **Example 4** by going to the **File** menu and choosing **Open Example Template**.

Option	Value
Design Tab	
Solve For	**Sample Size**
Power	**0.9**
Alpha	**0.05**
G (Number of Groups)	**2**
Group Allocation Input Type	**Equal (n1 = ⋯ = nG)**
μ0 Input Type	**Enter μ0 (Group Means|H0)**
μ0 (Group Means|H0)	**0 10**
μ1 Input Type	**Enter μ1 (Group Means|H1)**
μ1 (Group Means|H1)	**0 2**
σ (Standard Deviation)	**100**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Numeric Results

Number of Groups 2

	Sample Size		Group Means H0		Group Means H1		Std	SD of Standardized Means		
	Total	Grp	Means	SD(μ0)	Means	SD(μ1)	Dev	H0	H1	
Power	N	ni	μ0	σm0	μ1	σm1	σ	f0	f1	Alpha
0.9	5414	2707	μ0(1)	5	μ1(1)	1	100	0.05	0.01	0.05

Value Lists

Name	Value
μ0(1)	0, 10
μ1(1)	0, 2

Group Sample Size Details

n	N	Group Sample Sizes	Group Allocation Proportions
n(1)	5414	2707, 2707	0.5, 0.5

PASS also found N = 5414. So, the two procedures are in perfect agreement.