

Chapter 601

Hotelling's One-Sample T^2

Introduction

This module calculates power for Hotelling's one-group T-squared (T^2) test statistic. Hotelling's One-Sample T^2 is an extension of the univariate one-sample T-test to the case where the number of response variables is greater than one.

Assumptions

The following assumptions are made when using Hotelling's T^2 to analyze one group of data.

1. The response variables are continuous.
2. The residuals follow the multivariate normal probability distribution with mean zero and constant variance-covariance matrix.
3. The subjects are independent.

Technical Details

The formulas used to perform a Hotelling's T^2 power analysis provide exact answers if the above assumptions are met. These formulas can be found in many places. We use the results in Rencher (1998). We refer you to that reference for more details.

Test Statistic

In one-group case, a set of N observations is available on p response variables. We assume that all N observations have the same multivariate normal distribution with mean vector μ and variance-covariance matrix Σ and that Hotelling's T^2 is used for testing the null hypothesis that $\mu = \mu_0$ versus the alternative that $\mu \neq \mu_0$ where at least one component of μ is different from the corresponding component of μ_0 . Usually, μ_0 is a vector of zeros.

The value of T^2 is computed using the formula

$$T_{p,N-1}^2 = N(\bar{y} - \mu_0)'S^{-1}(\bar{y} - \mu_0)$$

where \bar{y} is the vector of sample means and S is the sample variance-covariance matrix.

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Power Calculation

To calculate power we need the non-centrality parameter for this distribution. With μ_1 as a vector of actual means, this non-centrality parameter is defined as follows

$$\begin{aligned}\lambda &= N(\mu_1 - \mu_0)' \Sigma^{-1} (\mu_1 - \mu_0) \\ &= N\Delta^2\end{aligned}$$

where

$$\Delta = \sqrt{(\mu_1 - \mu_0)' \Sigma^{-1} (\mu_1 - \mu_0)}$$

We define Δ as the *effect size* because it provides an expression for the magnitude of the standardized difference between the null and alternative means.

Using this non-centrality parameter, the power of the Hotelling's T² may be calculated for any values of the means and standard deviations. Since there is a simple relationship between the non-central T² and the non-central F , calculations are actually based on the non-central F using the formula

$$\text{Power} = 1 - \Pr(F' < F'_{\alpha, df_1, df_2, \lambda})$$

where

$$df_1 = p$$

$$df_2 = N - p$$

Procedure Options

This section describes the options that are specific to this procedure. These are located on the Design and Covariance tabs. For more information about the options of other tabs, go to the Procedure Window chapter.

Design Tab

The Design tab contains many of the options that you will be primarily concerned with.

Solve For

Solve For

This option specifies the parameter to be solved for.

When you choose to solve for *Sample Size*, the program searches for the lowest sample size that meets the alpha and power criterion you have specified.

Power and Alpha

Power

This option specifies one or more values for power. Power is the probability of rejecting a false null hypothesis, and is equal to one minus Beta. Beta is the probability of a type-II error, which occurs when a false null hypothesis is not rejected. In this procedure, a type-II error occurs when you fail to reject the null hypothesis of equal means when in fact the means are different.

Values must be between zero and one. Historically, the value of 0.80 (Beta = 0.20) was used for power. Now, 0.90 (Beta = 0.10) is also commonly used. A single value may be entered here or a range of values such as *0.8 to 0.95 by 0.05* may be entered.

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Alpha

This option specifies one or more values for the probability of a type-I error. A type-I error occurs when a true null hypothesis is rejected. In this procedure, a type-I error occurs when you reject the null hypothesis of equal means when in fact the means are equal.

Values must be between zero and one. Historically, the value of 0.05 has been used for alpha. This means that about one test in twenty will falsely reject the null hypothesis. You should pick a value for alpha that represents the risk of a type-I error you are willing to take in your experimental situation.

You may enter a range of values such as *0.01 0.05 0.10* or *0.01 to 0.10 by 0.01*.

Sample Size

Sample Size

Enter a value (or range of values) for the number of individuals.

Effect Size – Response Variables

Number of Response Variables

Enter the number of response (dependent or Y) variables. For a true multivariate test, this value will be greater than one.

The number of mean differences entered in the List of Mean Differences box or in the Column Containing the Mean Differences must be equal to this value. If you input the variance-covariance matrix from the spreadsheet, the number of columns specified must equal this value.

Effect Size – Mean Differences

Mean Difference Input Type

Specify the method used to input the vector of differences between the actual and null-hypothesized means, $\mu_1 - \mu_0$.

- **List of Mean Differences**
Select this option to enter a list of mean differences separated by spaces or commas.
- **Mean Differences in Spreadsheet**
Select this option to enter the mean differences in the spreadsheet. This is useful when the number of response variables is large.

You can view the input spreadsheet by pressing the spreadsheet button to the right.

List of Mean Differences

Enter a list of values representing the vector of differences between the actual and null-hypothesized means, $\mu_1 - \mu_0$. The values entered here represent the differences that you want to be able to detect.

Note that the number of values in the list must be equal to the number of response variables.

Column Containing the Mean Differences

Use this option to specify the spreadsheet column containing the vector of differences between the actual and null-hypothesized means, $\mu_1 - \mu_0$.

The response variables are represented down the rows. The number of rows with data in this column must equal the number of response variables.

You can view the input spreadsheet by pressing the spreadsheet button to the right.

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Effect Size – Means Multiplier

K (Means Multiplier)

Each of these values is multiplied times the means to give a range of effect sizes. A separate power calculation is generated for each value.

For example, if the original set of mean differences is “0 1 2”, setting this option to “1 2” would result in two sets of mean differences used in separate analyses: “0 1 2” and “0 2 4”.

If you want to ignore this setting, enter “1”.

Covariance Tab

This tab specifies the variance-covariance matrix.

Variance-Covariance Matrix

Variance-Covariance Matrix Input Type

Specify the method used to define the covariance matrix.

- **Standard Deviation and Correlation**

This option generates a variance-covariance matrix based on the standard deviations (σ), correlation (ρ), and correlation pattern specified below.

- **Variance-Covariance Matrix in spreadsheet**

The variance-covariance matrix is read in from the columns of the spreadsheet. This is the most flexible method, but specifying a variance-covariance matrix can be tedious. You will usually only use this method when a specific variance-covariance matrix is given to you.

You can view the input spreadsheet by pressing the spreadsheet button to the right.

Variance-Covariance Matrix – Standard Deviation

The parameters in this section provide a flexible way to specify Σ , the variance-covariance matrix. Because the variance-covariance matrix is symmetric, it can be represented as

$$\begin{aligned} \Sigma &= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} & \cdots & \sigma_1\sigma_p\rho_{1p} \\ \sigma_2\sigma_1\rho_{21} & \sigma_2^2 & \cdots & \sigma_2\sigma_p\rho_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_p\sigma_1\rho_{p1} & \sigma_p\sigma_2\rho_{p2} & \cdots & \sigma_p^2 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_p \end{bmatrix} \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{21} & 1 & \cdots & \rho_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{p1} & \rho_{p2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_p \end{bmatrix}. \end{aligned}$$

where p is the number of response variables.

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Thus, the covariance matrix can be represented with complete generality by specifying the standard deviations $\sigma_1, \sigma_2, \dots, \sigma_p$ and the correlation matrix

$$R = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{21} & 1 & \cdots & \rho_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{p1} & \rho_{p2} & \cdots & 1 \end{bmatrix}.$$

σ (Common Standard Deviation)

Specify a common standard deviation to be used for all response variables. The square of this value becomes the diagonal elements of the variance-covariance matrix. When this option is used, $\sigma_1 = \sigma_2 = \cdots = \sigma_p = \sigma$.

Since this is a standard deviation, it must be greater than zero.

ρ (Correlation)

Specify a correlation to be used in calculating the off-diagonal elements of the variance-covariance matrix.

Since this is a correlation, it must be between -1 and 1.

Correlation Pattern

This option specifies the pattern of the correlations in the variance-covariance matrix. Two options are possible:

- **Constant**

The value of ρ is used as the constant correlation. For example, if $\rho = 0.6$ and $p = 4$, the correlation matrix would appear as

$$R = \begin{bmatrix} 1 & 0.6 & 0.6 & 0.6 \\ 0.6 & 1 & 0.6 & 0.6 \\ 0.6 & 0.6 & 1 & 0.6 \\ 0.6 & 0.6 & 0.6 & 1 \end{bmatrix}$$

- **1st-Order Autocorrelation**

The value of ρ is used to generate a first-order autocorrelation pattern. This pattern reduces the autocorrelation at each successive step by multiplying the value at the last step by ρ . Thus, the pattern is $\rho \rho \times \rho \rho \times \rho \rho$ etc.

For example, $\rho = 0.6$ and $p = 4$, the correlation matrix would appear as

$$R = \begin{bmatrix} 1 & 0.6 & 0.36 & 0.216 \\ 0.6 & 1 & 0.6 & 0.36 \\ 0.36 & 0.6 & 1 & 0.6 \\ 0.216 & 0.36 & 0.6 & 1 \end{bmatrix}$$

This pattern is often chosen as the most realistic when little is known about the correlation pattern and the responses variables are measured across time.

Variance-Covariance Matrix – Variance-Covariance Matrix in Spreadsheet

This option instructs the program to read the variance-covariance matrix from the spreadsheet.

Columns Containing the Variance-Covariance Matrix

This option designates the columns on the input spreadsheet holding the variance-covariance matrix. The number of columns and number of rows with data must match the number of response variables specified. The software will check to be certain that the matrix is positive definite.

Hotelling's One-Sample T²**Example 1 – Power and Validation using Rencher (1998)**

Rencher (1998) page 106 presents an example of power calculations for the one-group case in which the mean differences are both 1.88 such that

$$\mu_1 - \mu_0 = \begin{bmatrix} 1.88 \\ 1.88 \end{bmatrix}$$

and the variance-covariance matrix is

$$\Sigma = \begin{bmatrix} 56.78 & 11.98 \\ 11.98 & 29.28 \end{bmatrix}$$

When N is 25 and the significance level is 0.05, Rencher calculated the power to be 0.3397. To allow for a nice chart, we will calculate the power for several sample sizes and for K equal 1.0 and 1.5.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Hotelling's One-Sample T²** procedure window by expanding **Means**, then clicking on **Multivariate Means**, and then clicking on **Hotelling's One-Sample T²**. You may then make the appropriate entries as listed below, or open **Example 1a** by going to the **File** menu and choosing **Open Example Template**. You can see that the values have been loaded into the spreadsheet by clicking on the spreadsheet button.

<u>Option</u>	<u>Value</u>
Design Tab	
Solve For	Power
Alpha.....	0.05
Sample Size.....	5 15 25 35 50 75 100 150
Number of Response Variables.....	2
Mean Difference Input Type	List of Mean Differences
List of Mean Differences	1.88 1.88
K (Means Multiplier).....	1 1.5

Covariance Tab

Variance-Covariance Matrix Input Type**Variance-Covariance Matrix in Spreadsheet**
 Columns Containing the V-C Matrix**VC1-VC2**

Input Spreadsheet Data

Row	Diffs	VC1	VC2
1	1.88	56.78	11.98
2	1.88	11.98	29.28

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Annotated Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Report

Numeric Results

Mean Differences: 1.88, 1.88
 Variance-Covariance Matrix: Input Columns = VC1-VC2

Power	N	Means	Number	DF1	DF2	Effect	Alpha
		Multiplier	of Y's				
		K				Size	
0.07372	5	1.0	2	2	3	0.380	0.050
0.19962	15	1.0	2	2	13	0.380	0.050
0.33787	25	1.0	2	2	23	0.380	0.050
0.47067	35	1.0	2	2	33	0.380	0.050
0.64085	50	1.0	2	2	48	0.380	0.050
0.83108	75	1.0	2	2	73	0.380	0.050
0.92822	100	1.0	2	2	98	0.380	0.050
0.98954	150	1.0	2	2	148	0.380	0.050
0.10403	5	1.5	2	2	3	0.570	0.050
0.40330	15	1.5	2	2	13	0.570	0.050
0.66345	25	1.5	2	2	23	0.570	0.050
0.83020	35	1.5	2	2	33	0.570	0.050
0.94754	50	1.5	2	2	48	0.570	0.050
0.99435	75	1.5	2	2	73	0.570	0.050
0.99952	100	1.5	2	2	98	0.570	0.050
1.00000	150	1.5	2	2	148	0.570	0.050

Report Definitions

Power is the probability of rejecting a false null hypothesis. Note that Power = 1 - Beta.

N is the sample size, the number of subjects in the experiment or study.

K is a constant by which all means are multiplied.

DF1 is the first degrees of freedom for T². It is equal to the number of response variables (Y's).

DF2 is the second degrees of freedom for T².

Effect Size is a standardized version of T² under the alternative hypothesis.

Alpha is the probability of rejecting a true null hypothesis.

Summary Statements

For a one-sample Hotelling's T² test, a sample size of 5 achieves 7% power at a significance level of 0.050 to detect an effect size of 0.380, which represents the differences between the actual and null-hypothesized means of the 2 response variables (1.88, 1.88) adjusted by the variance-covariance matrix (Input Columns = VC1-VC2).

This report gives the power for each value of *N* and *K*. Notice that the power for *K* = 1 and *N* = 25 is 0.3379. This is slightly different than the 0.3397 obtained by interpolation by Rencher.

Means Matrix

Means Matrix

Name	Mean
Y1	1.880
Y2	1.880

This report shows the mean differences that were read in. When a Means Multiplier, *K*, is used, each value of *K* is multiplied times each of these values.

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Variance-Covariance Matrix

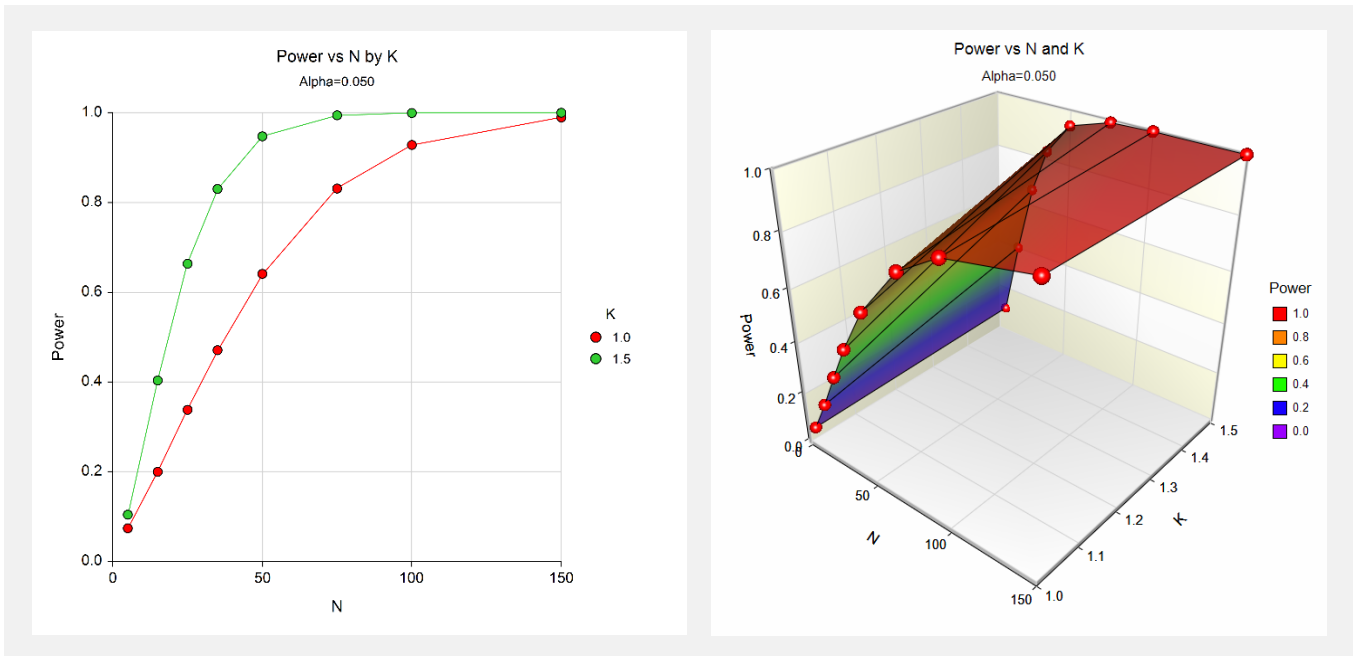
Variance-Covariance Matrix

Response	Y1	Y2
Y1	7.535	0.294
Y2	0.294	5.411

σ 's on the diagonal, ρ 's on the off-diagonal(s)

This report shows a modified variance-covariance matrix with σ 's on the diagonal, ρ 's on the off-diagonal(s). This matrix is created from the variance-covariance matrix that was read in from the spreadsheet or generated by the σ and ρ settings on the Covariance tab.

Chart Section



These charts show the relationship between power and N for each value of K .

The very same report can be generated by inputting the mean differences from the spreadsheet (Example 1b):

<u>Option</u>	<u>Value</u>
Design Tab	
Mean Difference Input Type	Mean Differences in Spreadsheet
Column Containing the Mean Differences.....	DIFFS

Input Spreadsheet Data

Row	DiffS	VC1	VC2
1	1.88	56.78	11.98
2	1.88	11.98	29.28