Chapter 136

Non-Unity Null Tests for the Ratio of Two Variances

Introduction

This procedure calculates power and sample size of inequality tests of (total = between + within) variances from a two-group, parallel design for the case when the ratio assumed by the null hypothesis is not necessarily equal to one. This routine deals with the case in which the statistical hypotheses are expressed in terms of the ratio of the variances.

Technical Details

This procedure uses the formulation given in Chow, Shao, Wang, and Lokhnygina (2018), pages 217 - 220.

Suppose x_{ij} is the response of the i^{th} group (*i* = 1, 2) and j^{th} subject (*j* = 1, ..., *Ni*). The model analyzed in this procedure is

$$x_{ijk} = \mu_i + e_{ij}$$

where μ_i is the treatment effect and e_{ij} is the between-subject error term which is normally distributed with mean 0 and variance $V_i = \sigma_{Bi}^2$. Unbiased estimators of these variances are given by

$$\hat{V}_i = \frac{1}{N_i - 1}\sum_{j=1}^{N_i}\left(x_{ij} - \bar{x}_i\right)^2$$

$$\bar{x}_i = \frac{1}{N_i}\sum_{j=1}^{N_i} x_{ij}$$

A common test statistic to compare variabilities in the two groups is $T = \hat{V}_1/\hat{V}_2$. Under the usual normality assumptions, *T* is distributed as an *F* distribution with degrees of freedom $N_1 - 1$ and $N_2 - 1$.

Testing Variance Inequality with a Non-Unity Null

The following three sets of statistical hypotheses are used to test for variance inequality with a non-unity null

$$H_0: \sigma_1^2/\sigma_2^2 \geq R0 \quad \text{versus} \quad H_1: \sigma_1^2/\sigma_2^2 < R0,$$

$$H_0: \sigma_1^2/\sigma_2^2 \leq R0 \quad \text{versus} \quad H_1: \sigma_1^2/\sigma_2^2 > R0,$$

$$H_0: \sigma_1^2/\sigma_2^2 = R0 \quad \text{versus} \quad H_1: \sigma_1^2/\sigma_2^2 \neq R0,$$

where *R0* is the variance ratio assumed by the null hypothesis.

The corresponding test statistics are $T = (\hat{V}_1/\hat{V}_2)/R0$.

Power

The corresponding powers of these three tests are given by

$$\text{Power} = \text{P}\left(F < \left(\frac{R0}{R1}\right) F_{\alpha,N_1-1,N_2-1}\right)$$

$$\text{Power} = 1 - \text{P}\left(F < \left(\frac{R0}{R1}\right) F_{1-\alpha,N_1-1,N_2-1}\right)$$

$$\text{Power} = \text{P}\left(F < \left(\frac{R0}{R1}\right) F_{\alpha/2,N_1-1,N_2-1}\right) + 1 - \text{P}\left(F < \left(\frac{R0}{R1}\right) F_{1-\alpha/2,N_1-1,N_2-1}\right)$$

where *F* is the common F distribution with the indicated degrees of freedom, α is the significance level, and *R1* is the value of the variance ratio stated by the alternative hypothesis. Lower quantiles of F are used in the equation.

A simple binary search algorithm can be applied to this power function to obtain an estimate of the necessary sample size.

Example 1 – Finding Sample Size

A company has developed a generic drug for treating rheumatism and wants to compare it to the standard drug in terms of the variances. A parallel-group design is used to test the inequality using a two-sided test.

Company researchers set the variance ratio under the null hypothesis to 0.75, the significance level to 0.05, the power to 0.90, and the actual variance ratio values between 0.5 and 1.2. They want to investigate the range of required sample size values assuming that the two group sample sizes are equal.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For	**Sample Size**
Alternative Hypothesis	**Two-Sided (H1: σ²1/σ²2 ≠ R0)**
Power	**0.90**
Alpha	**0.05**
Group Allocation	**Equal (N1 = N2)**
R0 (H0 Variance Ratio)	**0.75**
R1 (Actual Variance Ratio)	**0.5 0.6 0.9 1 1.1 1.2**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Reports

Numeric Results

Solve For: Sample Size
Hypotheses: H0: $\sigma^2 1/\sigma^2 2$ = R0 vs. H1: $\sigma^2 1/\sigma^2 2 \neq$ R0

Power		Sample Size			Variance Ratio		
Target	Actual	N1	N2	N	H0 (Null) R0	Actual R1	Alpha
0.9	0.9004	258	258	516	0.75	0.5	0.05
0.9	0.9003	847	847	1694	0.75	0.6	0.05
0.9	0.9001	1267	1267	2534	0.75	0.9	0.05
0.9	0.9001	510	510	1020	0.75	1.0	0.05
0.9	0.9005	289	289	578	0.75	1.1	0.05
0.9	0.9011	193	193	386	0.75	1.2	0.05

Target Power The desired power value entered in the procedure. Power is the probability of rejecting a false null hypothesis.
Actual Power The actual power achieved. Because N1 and N2 are discrete, this value is usually slightly larger than the target power.
N1 The number of subjects from group 1.
N2 The number of subjects from group 2.
N The total number of subjects. N = N1 + N2.
R0 The variance ratio used to define the null hypothesis, H0.
R1 The value of the variance ratio at which the power is calculated.
Alpha The probability of rejecting a true null hypothesis.

Summary Statements

A parallel two-group design will be used to test whether the variance ratio ($\sigma^2 1$ / $\sigma^2 2$ = σ^2Trt / σ^2Ctrl) is different from 0.75 (H0: $\sigma^2 1$ / $\sigma^2 2$ = 0.75 versus H1: $\sigma^2 1$ / $\sigma^2 2 \neq$ 0.75). The comparison will be made using a two-sided, two-sample, variance-ratio F-test, with a Type I error rate (α) of 0.05. To detect a variance ratio of 0.5 with 90% power, the number of subjects needed will be 258 in Group 1 (treatment), and 258 in Group 2 (control).

Dropout-Inflated Sample Size

	Sample Size			Dropout-Inflated Enrollment Sample Size			Expected Number of Dropouts		
Dropout Rate	**N1**	**N2**	**N**	**N1'**	**N2'**	**N'**	**D1**	**D2**	**D**
20%	258	258	516	323	323	646	65	65	130
20%	847	847	1694	1059	1059	2118	212	212	424
20%	1267	1267	2534	1584	1584	3168	317	317	634
20%	510	510	1020	638	638	1276	128	128	256
20%	289	289	578	362	362	724	73	73	146
20%	193	193	386	242	242	484	49	49	98

Dropout Rate The percentage of subjects (or items) that are expected to be lost at random during the course of the study and for whom no response data will be collected (i.e., will be treated as "missing"). Abbreviated as DR.
N1, N2, and N The evaluable sample sizes at which power is computed. If N1 and N2 subjects are evaluated out of the N1' and N2' subjects that are enrolled in the study, the design will achieve the stated power.
N1', N2', and N' The number of subjects that should be enrolled in the study in order to obtain N1, N2, and N evaluable subjects, based on the assumed dropout rate. After solving for N1 and N2, N1' and N2' are calculated by inflating N1 and N2 using the formulas N1' = N1 / (1 - DR) and N2' = N2 / (1 - DR), with N1' and N2' always rounded up. (See Julious, S.A. (2010) pages 52-53, or Chow, S.C., Shao, J., Wang, H., and Lokhnygina, Y. (2018) pages 32-33.)
D1, D2, and D The expected number of dropouts. D1 = N1' - N1, D2 = N2' - N2, and D = D1 + D2.

Dropout Summary Statements

Anticipating a 20% dropout rate, 323 subjects should be enrolled in Group 1, and 323 in Group 2, to obtain final group sample sizes of 258 and 258, respectively.

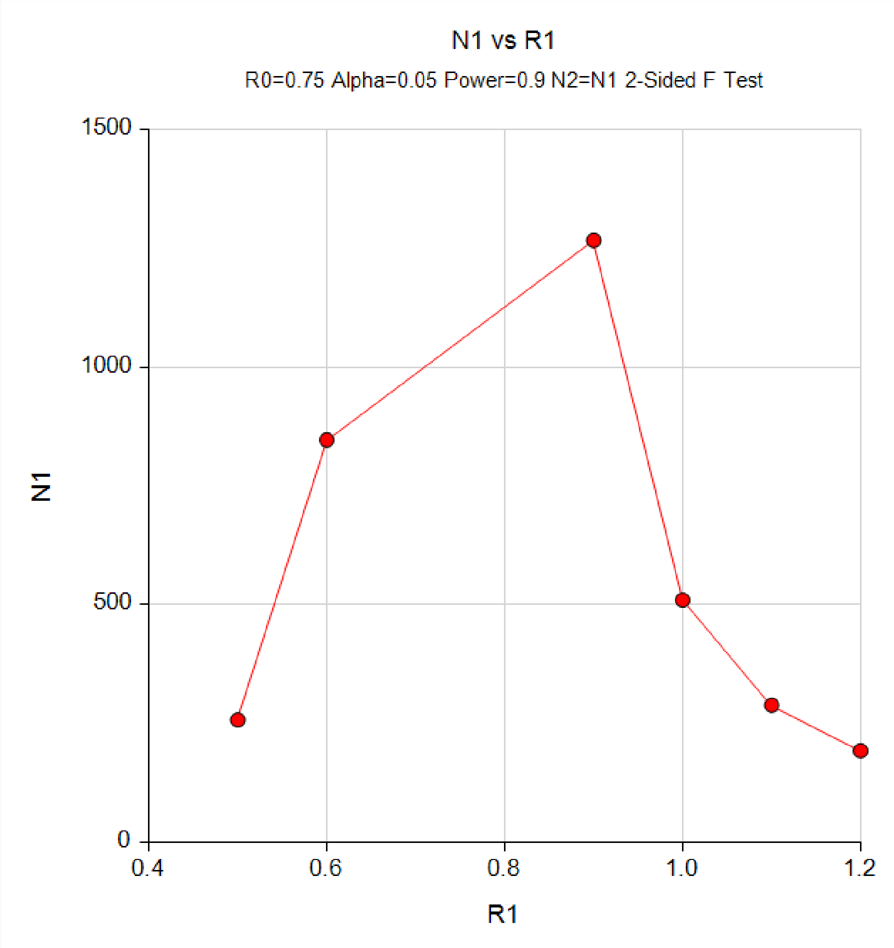
References

Johnson, N.L., Kotz, S., and Balakrishnan, N. 1995. Continuous Univariate Distributions, Volume 2, Second Edition. John Wiley & Sons. Hoboken, New Jersey.
Desu, M. M. and Raghavarao, D. 1990. Sample Size Methodology. Academic Press. New York.
Davies, Owen L. 1971. The Design and Analysis of Industrial Experiments. Hafner Publishing Company, New York.
Ostle, B. 1988. Statistics in Research. Fourth Edition. Iowa State Press. Ames, Iowa.
Zar, Jerrold H. 1984. Biostatistical Analysis (Second Edition). Prentice-Hall. Englewood Cliffs, New Jersey.
Chow, S.C., Shao, J., Wang, H., and Lokhnygina, Y. 2018. Sample Size Calculations in Clinical Research, Third Edition. Taylor & Francis/CRC. Boca Raton, Florida.
Chow, S.C., and Liu, J.P. 2014. Design and Analysis of Clinical Trials, Third Edition. John Wiley & Sons. Hoboken, New Jersey.

This report gives the sample sizes for the indicated scenarios.

Plots Section

Plots



These plots show the relationship between sample size and R1.

Example 2 – Validation using Chow et al. (2018)

The following example is shown in Chow *et al.* (2018) page 220.

Find the sample size when R0 is 1.21, the significance level to 0.05, the power is 0.8, and R1 is 0.53777778. They obtained N1 = N2 = 40. Their example is for a non-inferiority test.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For .... **Sample Size**
Alternative Hypothesis .... **One-Sided (H1: $\sigma^2 1/\sigma^2 2 < R0$)**
Power.. **0.80**
Alpha.. **0.05**
Group Allocation .... **Equal (N1 = N2)**
R0 (H0 Variance Ratio).................................. **1.21**
R1 (Actual Variance Ratio) **0.53777778**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Solve For: Sample Size
Hypotheses: H0: $\sigma^2 1/\sigma^2 2 \geq R0$ vs. H1: $\sigma^2 1/\sigma^2 2 < R0$

Power		Sample Size			Variance Ratio		
Target	Actual	N1	N2	N	H0 (Null) R0	Actual R1	Alpha
0.8	0.8051	40	40	80	1.21	0.538	0.05

The sample sizes match Chow et al. (2018).