

## Chapter 589

# One-Way Analysis of Variance Allowing Unequal Variances

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### Introduction

This procedure computes power and sample size of tests of the equality of multiple means which are analyzed using the Welch test which is recommended when the group variances are not equal. Note that the Welch test uses an adjusted degrees of freedom to compensate for the unequal variances. The results in this chapter come from Jan and Shieh (2014, 2019).

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### Technical Details

Suppose  $G$  groups each have a normal distribution and with means  $\mu_1, \mu_2, \dots, \mu_G$  and standard deviations  $\sigma_1, \sigma_2, \dots, \sigma_G$ . Let  $n_1, n_2, \dots, n_G$  denote the sample size of each group and let  $N$  denote the total sample size of all groups. The statistical hypotheses that are tested is

$$H_0: \omega^2 = 0 \text{ versus } H_1: \omega^2 > 0$$

Here we let  $\omega^2$  represent the variation in the standardized means.

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### Test Statistic

#### ANOVA F-Test

Assuming homogeneity of variance among the groups, the most popular procedure for analyzing a set of  $G$  means is the ANOVA F-Test which is calculated as follows.

$$F^* = \frac{SSM/(G - 1)}{SSE/(N - G)}$$

where  $SSM$  is the sum of squares of treatment means,  $SSE$  is the sum of squares of error, and  $N$  is the total sample size.

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### Welch's Test

If variance heterogeneity is suspected, a common approach is to use Welch's procedure which is calculated as follows.

$$W = \frac{\sum_{i=1}^G W_i (\bar{X}_i - \bar{X}) / (G - 1)}{1 + 2(G - 2)Q / (G^2 - 1)}$$

where

$$W_i = n_i / S_i^2, S_i^2 = \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2 / (n_i - 1), \bar{X}_i = \sum_{j=1}^{n_i} X_{ij} / n_i, \bar{X} = \sum_{i=1}^G W_i \bar{X}_i / U, U = \sum_{i=1}^G W_i, \text{ and}$$

$$Q = \sum_{i=1}^G \left(1 - \frac{W_i}{U}\right)^2 / (n_i - 1).$$

Under the null hypothesis, Welch (1951) gave the approximate distribution of  $W$  as an  $F$  distribution with degrees of freedom  $G - 1$  and  $\nu$ , where

$$\nu = \frac{G^2 - 1}{3Q}$$

### Jan and Shieh's Extension of Welch's Test

Jan and Shieh (2019) proposed testing the  $G$  group means in the face of variance heterogeneity using the hypotheses

$$H_0: \omega^2 = 0 \text{ versus } H_1: \omega^2 \geq 0$$

where

$$\omega^2 = \sum_{i=1}^G w_i (\mu_i - \mu^*)^2, w_i = \frac{n_i}{N\sigma_i^2}, \mu^* = \sum_{i=1}^G \frac{w_i \mu_i}{v}, \text{ and } v = \sum_{i=1}^G w_i.$$

Under  $H_0$ ,  $W$  is assumed to follow the  $F$  distribution  $W \sim F_{G-1, \nu}$ .

The null hypothesis is rejected at the significance level  $\alpha$  if  $W > F_{1-\alpha, G-1, \nu}$ .

## Power

The power function of the extended Welch's test computed at a particular set of means,  $\mu_{11}, \mu_{12}, \dots, \mu_{1G}$ , is given by

$$\text{Power} = \Pr[F'_{G-1, N-G, \Omega_1} > F_{1-\alpha, G-1, \eta}]$$

where

$$\Omega_1 = N\omega_1^2, \eta = (G^2 - 1)/(3\tau), \tau = \sum_{i=1}^G \left(1 - \frac{w_i}{v}\right)^2 / (n_i - 1), \omega_1^2 = \sum_{i=1}^G w_i (\mu_{1i} - \mu_1^*)^2, \text{ and}$$

$$\mu_1^* = \sum_{i=1}^G \frac{w_i \mu_{1i}}{v}.$$

When a sample size is desired, it can be determined using a standard binary search algorithm.

## Example 1 – Finding Sample Size

An experiment is being designed to assess sample size needed for an F test of the four means using the Welch test with a significance level of 0.05 and a power of 0.9. The first group is a control group and the three remaining groups are treatment groups. Previous studies have shown that the standard deviations in the four groups are {5, 4, 3, 4}. The sample sizes will be equal across all groups.

Three sets of alternative treatment means are to be compared: {17, 17, 13, 13}, {17, 16, 14, 13}, and {17, 15, 15, 13}. Note that these three alternative sets of means all have the same range (4) but different configurations.

### Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the procedure window. You may then make the appropriate entries as listed below, or open **Example 1** by going to the **File** menu and choosing **Open Example Template**.

<u>Option</u>	<u>Value</u>
<b>Design Tab</b>	
Solve For .....	<b>Sample Size</b>
Power .....	<b>0.90</b>
Alpha .....	<b>0.05</b>
G (Number of Groups) .....	<b>4</b>
Group Allocation Input Type .....	<b>Equal (n1 = ... = nG)</b>
$\mu_1$ Input Type .....	<b>Enter Columns Containing Sets of <math>\mu_1</math>'s</b>
Columns Containing Sets of $\mu_1$ 's .....	<b>1-3</b>
$\sigma$ Input Type .....	<b>Enter <math>\sigma</math> (Group Standard Deviations)</b>
$\sigma$ (Group Standard Deviations) .....	<b>5 4 3 4</b>

### Input Spreadsheet Data

Row	C1	C2	C3
1	17	17	17
2	17	16	15
3	13	14	15
4	13	13	13

### Annotated Output

Click the Calculate button to perform the calculations and generate the following output.

### Numeric Results

Numeric Results						
Number of Groups 4						
Power	– Sample Size –		Group Means H1 $\mu_1$	Std Dev $\sigma$	SD of Stand Means $\omega_1$	Alpha
	Total N	Grp ni				
0.90968	64	16	C1(1)	$\sigma(1)$	0.508	0.05
0.90619	112	28	C2(2)	$\sigma(1)$	0.371	0.05
0.90006	148	37	C3(3)	$\sigma(1)$	0.317	0.05

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## Value Lists

Name	Value
C1(1)	17, 17, 13, 13
C2(2)	17, 16, 14, 13
C3(3)	17, 15, 15, 13
$\sigma(1)$	5, 4, 3, 4

## Group Sample Size Details

n	N	Group Sample Sizes	Group Allocation Proportions
n(1)	64	16, 16, 16, 16	0.25, 0.25, 0.25, 0.25
n(2)	112	28, 28, 28, 28	0.25, 0.25, 0.25, 0.25
n(3)	148	37, 37, 37, 37	0.25, 0.25, 0.25, 0.25

## References

- Jan, S-L and Shieh, G. 2019. 'On the Extended Welch Test for Assessing Equivalence of Standardized Means'. Statistics in Biopharmaceutical Research. DOI:10.1080/19466315.2019.1654915
- Shieh, G. 2018. 'On Detecting a Minimal Important Difference among Standardized Means'. Current Psychology, Vol 37, Pages 640-647. Doi: 10.1007/s12144-016-9549-5
- Jan, S-L and Shieh, G. 2014. 'Sample size determinations for Welch's test in one-way heteroscedastic ANOVA'. British Journal of Mathematical and Statistical Psychology. Vol. 67, Pages 72-93.
- Shieh, G. and Jan, S-L. 2013. 'Determining Sample Size With a Given Range of Mean Effects in One-Way Heteroscedastic Analysis of Variance'. The Journal of Experimental Education, Vol 81(3), Pages 281-294.
- Cohen, Jacob. 1988. Statistical Power Analysis for the Behavioral Sciences. Lawrence Erlbaum Associates. Hillsdale, New Jersey.
- Welch, B.L. 1951. 'On the Comparison of Several Mean Values: An Alternative Approach'. Biometrika, 38, 330-336.

## Report Definitions

- Power is the probability of rejecting a false null hypothesis in favor of the alternative hypothesis.
- N, the Total Sample Size, is the total number of subjects in the study found by summing the group sample sizes.
- $\mu_1$ , the Group Means | H1, is the column name and set number of the group means under the alternative hypothesis. This is the set of means at which the power is calculated.
- $\sigma$ , the Group Standard Deviations, is the set name and number of the group standard deviations.
- $\omega_1$ , the SD of Stand Means, is the standard deviation of the standardized means ( $\mu_1/\sigma$ ) assumed by the alternative hypothesis, H1.
- Alpha is the significance level of the test: the probability of rejecting H0 when it is actually true.

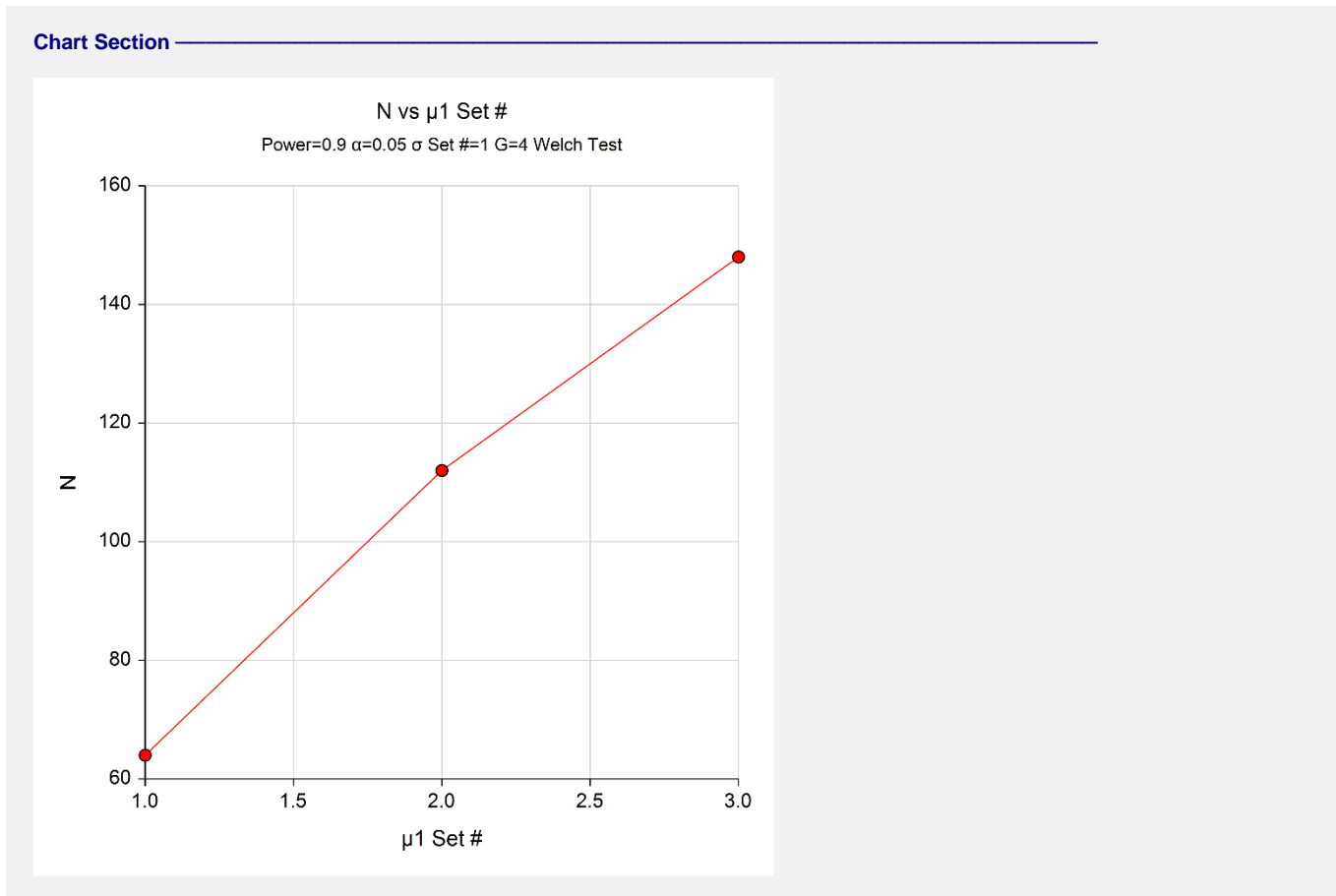
## Summary Statements

In a one-way ANOVA study that allows for unequal group variances, a sample of 64 subjects, divided among 4 groups, achieves a power of 91%. This power assumes the data will be analyzed with Welch's test with a significance level of 0.05. The group subject counts are 16, 16, 16, 16. The group means under the null hypothesis are assumed to be equal. The group means under the alternative hypothesis are 17, 17, 13, 13. The group standard deviations are 5, 4, 3, 4. The standard deviation of the standardized means under the alternative hypothesis is 0.508.

This report shows the numeric results of this study.

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## Chart Section



This plot gives a visual presentation of the results in the Numeric Report. Note that the horizontal axis plots the indices {1, 2, 3} of  $\mu_1(1)$ ,  $\mu_1(2)$ , and  $\mu_1(3)$ . That is, the horizontal value “1”, is the index of the set  $\mu_1(1)$ , and so on.

## Example 2 – Validation using Jan and Shieh (2014)

Jan and Shieh (2014) page 79 presents an example in which  $\alpha = 0.05$ ,  $G = 4$ , the sample sizes are  $\{10, 20, 30, 40\}$ , the standard deviations are  $\{1, 2, 3, 4\}$ , and the alternative means are  $\{1, 0, 0, -1\}$ . The resulting power is given as 0.7129.

### Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the procedure window. You may then make the appropriate entries as listed below, or open **Example 2** by going to the **File** menu and choosing **Open Example Template**.

<u>Option</u>	<u>Value</u>
<b>Design Tab</b>	
Solve For .....	<b>Power</b>
Alpha.....	<b>0.05</b>
G (Number of Groups).....	<b>4</b>
Group Allocation Input Type .....	<b>Enter n (Group Sample Sizes)</b>
n (Group Sample Sizes) .....	<b>10 20 30 40</b>
$\mu_1$ Input Type.....	<b>Enter <math>\mu_1</math> (Group Means H1)</b>
$\mu_1$ (Group Means H1).....	<b>1 0 0 -1</b>
$\sigma$ Input Type.....	<b>Enter <math>\sigma</math> (Group Standard Deviations)</b>
$\sigma$ (Group Standard Deviations).....	<b>1 2 3 4</b>

### Output

Click the Calculate button to perform the calculations and generate the following output.

#### Numeric Results

Numeric Results						
Number of Groups 4						
	— Sample Size —		Group	Std	SD of	
	Total	Grp	Means	Dev	Stand	
<b>Power</b>	<b>N</b>	<b>n</b>	<b>H1</b>	<b><math>\sigma</math></b>	<b><math>\omega_1</math></b>	<b>Alpha</b>
0.71286	100	n(1)	$\mu_1(1)$	$\sigma(1)$	0.313	0.05
<b>Value Lists</b>						
<b>Name</b>	<b>Value</b>					
$\mu_1(1)$	1, 0, 0, -1					
$\sigma(1)$	1, 2, 3, 4					
<b>Group Sample Size Details</b>						
<b>n</b>	<b>N</b>	<b>Group Sample Sizes</b>		<b>Group Allocation Proportions</b>		
n(1)	100	10, 20, 30, 40		0.1, 0.2, 0.3, 0.4		

PASS also found the power to be 0.71286. The procedure is validated.