Chapter 594

One-Way Analysis of Variance Contrasts Allowing Unequal Variances

Introduction

This procedure computes power and sample size of non-zero null tests of contrasts of multiple means which are analyzed using the Welch-Satterthwaite t-test. This method is recommended when the group variances are not equal. The results in this chapter come from Jan and Shieh (2016).

Technical Details

Suppose G groups each have a normal distribution and with means $\mu_1, \mu_2, \ldots, \mu_G$ and standard deviations $\sigma_1, \sigma_2, \ldots, \sigma_G$. Let $n_1, n_2, \ldots, n_G$ denote the sample size of each group and let N denote the total sample size of all groups. The non-zero null F-test is used to show that the means are significantly different from each other. Sometimes, however, it is of interest to test a specific comparison or contrast of the means. This procedure provides results for contrasts used in an ANOVA design.

A *comparison* is a weighted average of the means, in which some of the weights may be negative. When the weights sum to zero, the comparison is called a *contrast*.

Suppose you want to test whether a linear contrast of the means is significantly different from zero. This contrast is defined as

$$\delta = \sum_{i=1}^{G} c_i \mu_i$$

Here $c_1, c_2, \ldots, c_G$ are the contrast coefficients. Note that $\sum_{i=1}^{G} c_i = 0$ is required. Also, to make different contrasts comparable, Kirk (2013) suggests that $\sum_{i=1}^{G} |c_i| = 2$.

A unbiased estimate of δ is found by replacing the population means by the corresponding sample means.

The hypothesis testing of $\mathrm{H}_0: \delta = \delta_0$ versus $\mathrm{H}_1: \delta \neq \delta_0$ can be conducted using

$$T = \frac{\hat{\delta} - \delta_0}{\widehat{\sigma(\delta)}}$$

where

$$\delta_0 = \sum_{i=1}^{G} c_i \mu_{0i}$$

$$\widehat{\sigma^2(\delta)} = \sum_{i=1}^{G} c_i^2 S_i^2 / n_i$$

$$S_i^2 = \sum_{j=1}^{N_i} \left(X_{ij} - \bar{X}_i\right)^2 / (n_i - 1)$$

Here $\widehat{\sigma^2(\delta)}$ is the estimator of $\sigma^2(\delta) = \mathrm{Var}\left(\hat{\delta}\right)$ where

$$\sigma^2(\delta) = \sum_{i=1}^{G} c_i^2 \sigma_i^2 / n_i$$

Under the null hypothesis, Satterthwaite (1946) and Welch (1947) showed that *T* is approximately distributed as a Student's *t* with *v* degrees of freedom where

$$v = \frac{\left(\sum_{i=1}^{G} \frac{c_i^2 \sigma_i^2}{n_i}\right)^2}{\sum_{i=1}^{G} \frac{c_i^4 \sigma_i^4}{n_i^2 (n_i - 1)}}$$

This value is estimated as

$$\hat{v} = \frac{\left(\sum_{i=1}^{G} \frac{c_i^2 S_i^2}{n_i}\right)^2}{\sum_{i=1}^{G} \frac{c_i^4 S_i^4}{n_i^2 (n_i - 1)}}$$

Hence, the test statistic *T* has the approximate distribution

$$T \sim t_{\hat{v}}$$

The Welch-Satterthwaite test rejects H_0 at a significance level α if $|T| > t_{\hat{v},1-\frac{\alpha}{2}}$.

Power

Shieh and Jan (2015) noted that *T* has the general approximate distribution

$$T \sim t_{\hat{v},\Delta}$$

where $t_{\hat{v},\delta}$ is a noncentral *t* with $\hat{v}$ degrees of freedom and noncentrality parameter Δ. Here, Δ is defined as

$$\Delta = \frac{\delta_1 - \delta_0}{\sigma(\delta)}$$

where $\delta_1 = \sum_{i=1}^{G} c_i \mu_{1i}$.

Hence, the power can be approximated as

$$Power = P\left(\left|t_{v,\Delta}\right| > t_{v,1-\frac{\alpha}{2}}\right)$$

When a sample size is desired, it can be determined using a standard binary search algorithm.

Example 1 – Finding Sample Size

This example will show all the reports that are available in the procedure. It will also show the impact on sample size of changing various options.

Suppose an experiment is being designed to assess the sample size needed for a one-way design with 3 groups that will be analyzed with various contrasts using the extended Welch test at a significance level of 0.05 and a power of 0.9. The null means are all 0. The alternative means are {1, 2, 4}. The standard deviations are {1, 3, 4}.

In order to showcase the use of the spreadsheet input, we will fill it with data so that it appears as follows.

Eq	SD	Con1	Con2	Con3
1	1	-1	0.5	0.5
1	3	0.5	-1	0.5
1	4	0.5	0.5	-1

Note that we have changed the default column names to more descriptive names. The is easily accomplished be right clicking on the name. We changed C1 to Eq, C2 to SD, C3 to Con1, C4 to Con2, and C5 to Con3.

The first two columns will be used as group allocations patterns: "Eq" is for all groups equal and "SD" is for group sizes proportional to the standard deviations.

The last three columns hold various sets of contrast coefficients. The first compares group 1 to the average of groups 2 and 3. The second emphasizes group 2 and the third emphasizes group 3.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the procedure window. You may then make the appropriate entries as listed below, or open **Example 1** by going to the **File** menu and choosing **Open Example Template**.

Option	Value
Design Tab	
Solve For	**Sample Size**
Alternative Hypothesis	**Two-Sided (H1: δ ≠ δ0)**
Power	**0.90**
Alpha	**0.05**
G (Number of Groups)	**3**
Group Allocation Input Type	**Enter Columns of r's (Allocation Patterns)**
Columns of r's (Allocation Patterns)	**EQ-SD**
μ0 Input Type	**Enter μ0 (Group Means|H0)**
μ0 (Group Means|H0)	**0**
μ1 Input Type	**Enter μ1 (Group Means|H1)**
μ1 (Group Means|H1)	**1 2 4**
Contrast Input Type	**Multiple Lists of Contrast Coefficients**
Multiple Lists of Coefficients	**CON1-CON3**
σ Input Type	**Enter σ (Group Standard Deviations)**
σ (Group Standard Deviations)	**1 3 4**

Input Spreadsheet Data

Row	Eq	SD	Con1	Con2	Con3
1	1	1	-1.0	0.5	0.5
2	1	3	0.5	-1.0	0.5
3	1	4	0.5	0.5	-1.0

Annotated Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Numeric Results

Number of Groups 3
Hypotheses H0: δ = δ0 vs. H1: δ ≠ δ0

Contrast Coef C	Power	Sample Size Total N	Sample Size Alloc r	Sample Size Grp n	Means H0 μ0	Means H1 μ1	Contrast Among Means H0 δ0	Contrast Among Means H1 δ1	SD's σ	SE of δ's σ(δ)	NCP Δ	Alpha
Con1(1)	0.90158	60	Eq(1)	n(1)	μ0(1)	μ1(1)	0	2.0	σ(1)	0.602	3.322	0.05
Con2(2)	0.90043	1674	Eq(1)	n(2)	μ0(1)	μ1(1)	0	0.5	σ(1)	0.154	3.245	0.05
Con3(3)	0.90348	99	Eq(1)	n(3)	μ0(1)	μ1(1)	0	-2.5	σ(1)	0.749	-3.339	0.05
Con1(1)	0.91365	64	SD(2)	n(4)	μ0(1)	μ1(1)	0	2.0	σ(1)	0.586	3.411	0.05
Con2(2)	0.90046	1432	SD(2)	n(5)	μ0(1)	μ1(1)	0	0.5	σ(1)	0.154	3.245	0.05
Con3(3)	0.90837	72	SD(2)	n(6)	μ0(1)	μ1(1)	0	-2.5	σ(1)	0.745	-3.354	0.05

Value Lists

Name	Value
Con1(1)	-1, 0.5, 0.5
Con2(2)	0.5, -1, 0.5
Con3(3)	0.5, 0.5, -1
Eq(1)	1, 1, 1
SD(2)	1, 3, 4
μ0(1)	0, 0, 0
μ1(1)	1, 2, 4
σ(1)	1, 3, 4

Group Sample Size Details

n	N	Group Sample Sizes	Group Allocation Proportions
n(1)	60	20, 20, 20	0.333, 0.333, 0.333
n(2)	1674	558, 558, 558	0.333, 0.333, 0.333
n(3)	99	33, 33, 33	0.333, 0.333, 0.333
n(4)	64	8, 24, 32	0.125, 0.375, 0.5
n(5)	1432	179, 537, 716	0.125, 0.375, 0.5
n(6)	72	9, 27, 36	0.125, 0.375, 0.5

References

Jan, S.L. and Shieh, G. 2016. 'A systematic approach to designing statistically powerful heteroscedastic 2 x 2 factorial studies while minimizing financial costs.' BMC Medical Research Methodology, 16:114.
Kirk, Roger E. 2013. Experimental Design: Procedures for the Behavioral Sciences, 4th Edition. Sage. Washington, D.C.
Luh, W.M. and Guo, J.H. 2016. 'Allocating sample sizes to reduce budget for fixed-effect 2 x 2 heterogeneous analysis of variance.' Journal of Experimental Education, 84:197-211.
Satterthwaite, F.E. 1946. 'An approximate distribution of estimate of variance components,' Biometric Bulletin, 2:110-114.
Shieh, G. and Jan, S-L. 2015. 'Power and sample size calculations for testing linear combinations of group means under variance heterogeneity with applications to meta and moderation analysis'. Psicologica, 36:367-390.
Welch, B.L. 1951. 'On the Comparison of Several Mean Values: An Alternative Approach'. Biometrika,38,330-336.

Report Definitions

C, the Contrast Coefficients, is the name of the set containing the contrast coefficients. The only restriction is that the sum of the coefficients must be zero and $\delta_0 \neq \delta_1$.

Power is the probability of rejecting a false non-zero null hypothesis in favor of the alternative hypothesis.

N, the Total Sample Size, is the total number of subjects in the study found by summing the group sample sizes.

r, the Group Allocation, is the name and number of the set containing the Group Allocation Pattern, r. These values are then rescaled so they sum to one to form the Group Allocation Proportions.

n, the Group Sample Size, is the name and number of the set containing the sample size of each group.

μ_0, the Group Means | H0, is the name and number of the set containing the group means under the null hypothesis. Note that $\delta_0 = \mu_0'C$.

μ_1, the Group Means | H1, is the name and number of the set containing the group means under the alternative hypothesis. This is the set of means at which the power is calculated using $\delta_1 = \mu_1'C$.

δ_0, the Contrast Among Means | H0 or $\mu_0'C$, is the dot product of μ_0 and C assumed by H0. The dot product is the sum of the products of the corresponding entries of the two sets of numbers. Note that you must have $\delta_0 \neq \delta_1$.

δ_1, the Contrast Among Means | H1 or $\mu_1'C$, is the dot product of μ_1 and C assumed by H1. The dot product is the sum of the products of the corresponding entries of the two sets of numbers. Note that you must have $\delta_0 \neq \delta_1$.

σ, the Group SD's, is the name and number of the set containing the standard deviation of each group.

$\sigma(\delta)$, the standard error of the δ's, is used in the calculation of Δ. Note that $\sigma(\delta)^2 = \Sigma[(C_j \sigma_j)^2 / n_j]$.

Δ, the NCP (noncentrality parameter), is used with the noncentral t-distribution to calculate the power. Note that $\Delta = (\delta_1 - \delta_0) / \sigma(\delta)$.

Alpha is the significance level of the test: the probability of rejecting H0 when it is actually true.

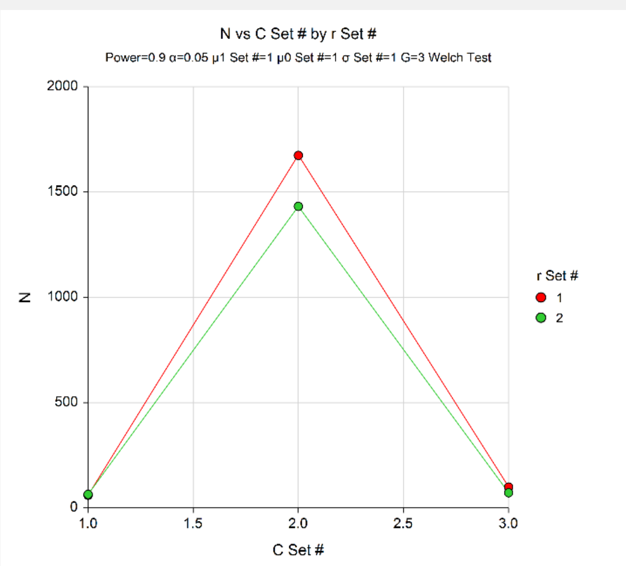
Summary Statements

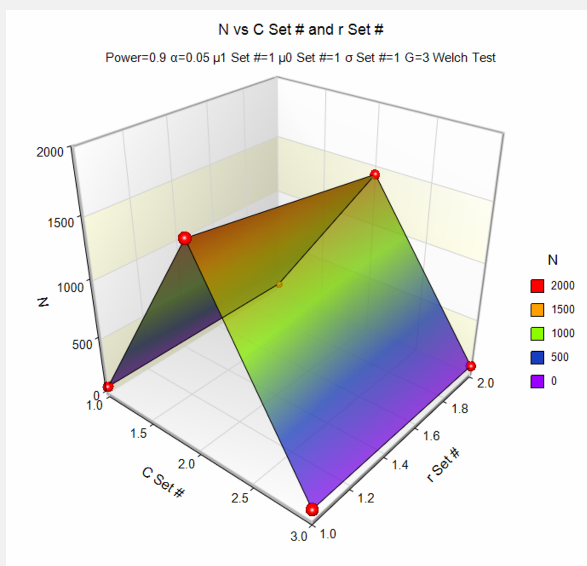
In non-zero null, one-way ANOVA study that allows for unequal group variances, a sample of 60 subjects, divided among 3 groups, achieves a power of 90%. This power assumes the data will be analyzed with Welch's adjusted degrees of freedom t-test with a significance level of 0.05. The group subject counts are 20, 20, 20. The group means under the null hypothesis are 0, 0, 0. The group means under the alternative hypothesis are 1, 2, 4. The group standard deviations are 1, 3, 4. The value of the contrast applied to the means under the null hypothesis is 0. The value of the contrast applied to the means under the alternative hypothesis is 2. The noncentrality parameter is 3.322.

This report shows the numeric results of this study.

Chart Section

Chart Section





The plots give a visual presentation of the results in the Numeric Report.

The main impression conveyed by this report and plot is that the required sample size is heavily impacted by the choice of the contrast coefficients, and, to a lesser degree, by the allocation pattern.

Example 2 – Validation using Jan and Shieh (2016)

Jan and Shieh (2016) page 6, Table 1, presents an example in which alpha = 0.05, G = 4, the sample sizes are {16, 14, 7, 15}, the standard deviations are {0.83, 0.72, 0.34, 0.77}, the null means are {0, 0, 0, 0}, and the alternative means are {1.23, 0.42, 0.13, 0.38}. The contrast coefficients are {0.5, -0.5, -0.5, 0.5}. The resulting power is given as 0.8038.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the procedure window. You may then make the appropriate entries as listed below, or open **Example 2** by going to the **File** menu and choosing **Open Example Template**.

Option	Value
Design Tab	
Solve For	**Power**
Alternative Hypothesis	**Two-Sided (H1: δ ≠ δ0)**
Alpha	**0.05**
G (Number of Groups)	**4**
Group Allocation Input Type	**Enter n (Group Sample Sizes)**
n (Group Sample Sizes)	**16 14 7 15**
μ0 Input Type	**Enter μ0 (Group Means|H0)**
μ0 (Group Means|H0)	**0**
μ1 Input Type	**Enter μ1 (Group Means|H1)**
μ1 (Group Means|H1)	**1.23 0.42 0.13 0.38**
Contrast Input Type	**List of Contrast Coefficients**
Contrast Coefficients	**0.5 -0.5 -0.5 0.5**
σ Input Type	**Enter σ (Group Standard Deviations)**
σ (Group Standard Deviations)	**0.83 0.72 0.34 0.77**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Numeric Results

Number of Groups 4
Hypotheses H0: δ = δ0 vs. H1: δ ≠ δ0

Contrast Coef C	Power	Sample Size Total N	Sample Size Grp n	Means H0 μ0	Means H1 μ1	Contrast Among Means H0 δ0	Contrast Among Means H1 δ1	SD's σ	SE of δ's σ(δ)	NCP Δ	Alpha
C(1)	0.80376	52	n(1)	μ0(1)	μ1(1)	0	0.53	σ(1)	0.184	2.873	0.05

Value Lists

Name	Value
C(1)	0.5, -0.5, -0.5, 0.5
n(1)	16, 14, 7, 15
μ0(1)	0, 0, 0, 0
μ1(1)	1.23, 0.42, 0.13, 0.38
σ(1)	0.83, 0.72, 0.34, 0.77

Group Sample Size Details

n	N	Group Sample Sizes	Group Allocation Proportions
n(1)	52	16, 14, 7, 15	0.308, 0.269, 0.135, 0.288

PASS also found the power to be 0.80376 which rounds to 0.8038. The procedure is validated.