

## Chapter 802

# Pearson's Correlation Tests (Simulation)

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## Introduction

This procedure analyzes the power and significance level of the Pearson Product-Moment Correlation Coefficient significance test using Monte Carlo simulation. This test is used to test whether the correlation is equal to a specified value. For each scenario that is set up, two simulations are run. One simulation estimates the significance level and the other estimates the power.

The Pearson correlation coefficient,  $\rho$  (rho), is a popular statistic for describing the strength of the relationship between two variables. It is the slope of the regression line between two variables when both variables have been standardized by subtracting their means and dividing by their standard deviations. The correlation ranges between plus and minus one. The population correlation  $\rho$  is estimated by the sample correlation coefficient  $r$ .

When  $\rho$  is used as a descriptive statistic, no special distributional assumptions need to be made about the variables ( $Y$  and  $X$ ) from which it is calculated. When hypothesis tests are made, you assume that the observations are independent and that the variables are distributed according to the bivariate-normal density function. However, as with the t-test, tests based on the correlation coefficient are robust to moderate departures from this normality assumption.

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## Difference Between Linear Regression and Correlation

The correlation coefficient is used when both  $X$  and  $Y$  are normally distributed (in fact, the assumption actually is that  $X$  and  $Y$  follow a bivariate normal distribution). In the linear regression context, no statement is made about the distribution of  $X$ . In fact,  $X$  is not even a random variable. Instead, it is a set of fixed values such as 10, 20, 30 or -1, 0, 1. Because of this difference in definition, we have included both Linear Regression and Correlation algorithms. This module deals with the Correlation (random  $X$ ) case.

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## Technical Details

*Computer simulation* allows us to estimate the power and significance level that is actually achieved by a test procedure in situations that are not mathematically tractable. Computer simulation was once limited to mainframe computers. But, in recent years, as computer speeds have increased, simulation studies can be completed on desktop and laptop computers in a reasonable period of time.

The steps to a simulation study are

1. Specify the test procedure and the test statistic. This includes the significance level, sample size, and underlying data distributions.
2. Generate a random sample of points ( $X, Y$ ) from the bivariate distribution specified by the alternative hypothesis. Calculate the test statistic from the simulated data and determine if the null hypothesis is accepted or rejected. These samples are used to calculate the power of the test. In the case of paired

## Pearson's Correlation Tests (Simulation)

data, the individual values are simulated are constructed so that they exhibit the specified amount of correlation.

3. Generate a second random sample of points (X, Y) from the bivariate distribution specified by the null hypothesis. Calculate the test statistic from the simulated data and determine if the null hypothesis is accepted or rejected. These samples are used to calculate the significance-level of the test. In the case of paired data, the individual values are simulated so that they exhibit the specified amount of correlation.
4. Repeat steps 2 and 3 several thousand times, tabulating the number of times the simulated data leads to a rejection of the null hypothesis. The power is the proportion of simulated samples in step 2 that lead to rejection. The significance level is the proportion of simulated samples in step 3 that lead to rejection.

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## Simulating Paired Distributions

In this routine, paired data may be generated from the bivariate normal distribution or from two specified marginal distributions. In the latter case, the simulation should mimic the actual data generation process as closely as possible, including the marginal distributions and the correlation between the two variables.

Obtaining paired samples from arbitrary distributions with a set correlation is difficult because the joint, bivariate distribution must be specified and simulated. Rather than specify the bivariate distribution, **PASS** requires the specification of the two marginal distributions and the correlation between them.

Monte Carlo samples with given marginal distributions and correlation are generated using the method suggested by Gentle (1998). The method begins by generating a large population of random numbers from the two distributions. Each of these populations is evaluated to determine if their means are within a small relative tolerance (0.0001) of the target mean. If the actual mean is not within the tolerance of the target mean, individual members of the population are replaced with new random numbers if the new random number moves the mean towards its target. Only a few hundred such swaps are required to bring the actual mean to within tolerance of the target mean.

The next step is to obtain the target correlation. This is accomplished by permuting one of the populations until they have the desired correlation.

The above steps provide a large pool (ten thousand to one million items) of random number pairs that exhibit the desired characteristics. This pool is then sampled at random using the uniform distribution to obtain the random number pairs used in the simulation.

This algorithm may be stated as follows.

1. Draw individual samples of size M from the two distributions where M is a large number over 10,000. Adjust these samples so that they have the specified mean and standard deviation. Label these samples A and B. Create an index of the values of A and B according to the order in which they are generated. Thus, the first value of A and the first value of B are indexed as one, the second values of A and B are indexed as two, and so on up to the final set which is indexed as M.
2. Compute the correlation between the two generated variates.
3. If the computed correlation is within a small tolerance (usually less than 0.001) of the specified correlation, go to step 7.
4. Select two indices (I and J) at random using uniform random numbers.

## Pearson's Correlation Tests (Simulation)

5. Determine what will happen to the correlation if  $B_I$  is swapped with  $B_J$ . If the swap will result in a correlation that is closer to the target value, swap the indices and proceed to step 6. Otherwise, go to step 4.
6. If the computed correlation is within the desired tolerance of the target correlation, go to step 7. Otherwise, go to step 4.
7. End with a population with the required marginal distributions and correlation.

A population created by this procedure tends to exhibit more variation in the tails of the distribution than in the center. Hence, the results using the bivariate normal option will be slightly different from those obtained when a custom model with two normal distributions is used.

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## Test Statistic

This section describes the test statistic that is used.

### Sample Pearson Product-Moment Correlation Coefficient

The Pearson product-moment correlation coefficient is calculated from a sample of  $N$  data pairs  $(X, Y)$  using

$$r = \frac{\sum(X - \bar{X})(Y - \bar{Y})}{\sqrt{\sum(X - \bar{X})^2 \sum(Y - \bar{Y})^2}}$$

The distribution of  $r$  assuming  $X$  and  $Y$  follow the bivariate normal distribution is known and available in **PASS**. It can be used to determine critical values for  $r$ .

### Test Procedure

The testing procedure is as follows.  $H_0$  is the null hypothesis that the true correlation is a specific value,  $\rho_0$  (usually,  $\rho_0 = 0$ ).  $H_1$  represents the alternative hypothesis that the actual correlation of the population is  $\rho_1$ , which is not equal to  $\rho_0$ . Choose a value  $r_\alpha$ , based on the distribution of the sample correlation coefficient, so that the probability of rejecting  $H_0$  when  $H_0$  is true is equal to a specified value,  $\alpha$ .

Select a sample of  $N$  items from the population and compute the sample correlation coefficient,  $r_s$ . If  $r_s > r_\alpha$  reject the null hypothesis that  $\rho = \rho_0$  in favor of an alternative hypothesis that  $\rho = \rho_1$ , where  $\rho_1 > \rho_0$ . The power is the probability of rejecting  $H_0$  when the true correlation is  $\rho_1$ .

All calculations are based on the algorithm described by Guenther (1977) for calculating the cumulative correlation coefficient distribution.

## Example 1 – Finding the Power

Suppose a study will be run to test whether the correlation between forced vital capacity (X) and forced expiratory value (Y) in a particular population is non-zero. Find the power when  $\rho_0 = 0$ ,  $\rho_1 = 0.20$  and  $0.30$ , alpha is  $0.05$ , and  $N = 20, 60, 100$ .

### Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

#### Design Tab

Solve For .....	<b>Power</b>
Alternative Hypothesis .....	<b><math>\rho_1 \neq \rho_0</math></b>
Simulations .....	<b>5000</b>
Random Seed .....	<b>95994102</b> (for Reproducibility)
Alpha.....	<b>0.05</b>
N (Sample Size).....	<b>20 40 100</b>
$\rho_0$ (Correlation H0) .....	<b>0.0</b>
$\rho_1$ (Correlation H1) .....	<b>0.2 0.3</b>
Distribution.....	<b>Bivariate Normal</b>

### Output

Click the Calculate button to perform the calculations and generate the following output.

### Numeric Reports

#### Simulation Summary

Variables	Simulation Distribution	
	Power (H1)	Alpha (H0)
X and Y	Bivariate Normal	Bivariate Normal
Random Number Pool Size	10000	
Number of Simulations	5000	
Random Seed	95994102 (User-Entered)	
Run Time	1.35 seconds	

Pearson's Correlation Tests (Simulation)

**Numeric Results**

Solve For: **Power**  
 Correlation Hypotheses:  $H_0: \rho_1 = \rho_0$  vs.  $H_1: \rho_1 \neq \rho_0$

Scenario	Sample Size N	Power	Correlation		Alpha	
			$\rho_0$	$\rho_1$	Target	Actual
1	20	0.136	0	0.2	0.05	0.045
2	40	0.244	0	0.2	0.05	0.050
3	100	0.578	0	0.2	0.05	0.049
4	20	0.253	0	0.3	0.05	0.049
5	40	0.458	0	0.3	0.05	0.052
6	100	0.880	0	0.3	0.05	0.041

N The size of the sample drawn from the population. It is the number of X-Y data points in a sample.  
 Power The probability of rejecting a false null hypothesis. It is calculated by the power simulation.  
 $\rho_0$  The Pearson correlation coefficient assuming the null hypothesis,  $H_0$ . This is the value being tested.  
 $\rho_1$  The Pearson correlation coefficient assuming the alternative hypothesis,  $H_1$ . This is the value at which the power is computed.  
 Target Alpha The probability of rejecting a true null hypothesis. It is set by the user.  
 Actual Alpha The alpha level that was actually achieved by the experiment. It is calculated by the alpha simulation.

**Summary Statements**

A correlation (single group, Y versus X) design will be used to test whether the Pearson correlation is different from 0 ( $H_0: \rho = 0$  versus  $H_1: \rho \neq 0$ ). The comparison will be made using a two-sided, one-sample Pearson correlation test, with a Type I error rate ( $\alpha$ ) of 0.05. Based on 5000 simulations of the bivariate normal distribution (under the alternative hypothesis), to detect a (Pearson) correlation of 0.2 (corresponding to a difference of 0.2), with a sample size of 20, the power is 0.136.

**Power and Alpha Confidence Intervals from Simulations**

Scenario	Sample Size N	Power	95% Confidence Interval Limits for Power		Alpha		95% Confidence Interval Limits for Alpha	
			Lower	Upper	Target	Actual	Lower	Upper
1	20	0.136	0.126	0.145	0.05	0.045	0.039	0.051
2	40	0.244	0.232	0.256	0.05	0.050	0.044	0.056
3	100	0.578	0.565	0.592	0.05	0.049	0.043	0.055
4	20	0.253	0.241	0.265	0.05	0.049	0.043	0.055
5	40	0.458	0.444	0.472	0.05	0.052	0.045	0.058
6	100	0.880	0.871	0.889	0.05	0.041	0.036	0.046

N The size of the sample drawn from the population. It is the number of X-Y data points in a sample.  
 Power The probability of rejecting  $H_0$  when it is false. This is the actual value calculated by the power simulation.  
 Lower and Upper Limits of 95% C.I. of Power The limits of an exact, 95% confidence interval for power based on the binomial distribution. They are calculated from the power simulation.  
 Target Alpha The desired probability of rejecting a true null hypothesis at which the tests were run.  
 Actual Alpha The alpha achieved by the test as calculated by the alpha simulation.  
 Lower and Upper Limits of 95% C.I. of Alpha The limits of an exact, 95% confidence interval for alpha based on the binomial distribution. They are calculated from the alpha simulation.

## Pearson's Correlation Tests (Simulation)

**Dropout-Inflated Sample Size**

Dropout Rate	Sample Size N	Dropout- Inflated Enrollment Sample Size N'	Expected Number of Dropouts D
20%	20	25	5
20%	40	50	10
20%	100	125	25

Dropout Rate	The percentage of subjects (or items) that are expected to be lost at random during the course of the study and for whom no response data will be collected (i.e., will be treated as "missing"). Abbreviated as DR.
N	The evaluable sample size at which power is computed (as entered by the user). If N subjects are evaluated out of the N' subjects that are enrolled in the study, the design will achieve the stated power.
N'	The total number of subjects that should be enrolled in the study in order to obtain N evaluable subjects, based on the assumed dropout rate. N' is calculated by inflating N using the formula $N' = N / (1 - DR)$ , with N' always rounded up. (See Julious, S.A. (2010) pages 52-53, or Chow, S.C., Shao, J., Wang, H., and Lohknygina, Y. (2018) pages 32-33.)
D	The expected number of dropouts. $D = N' - N$ .

**Dropout Summary Statements**

Anticipating a 20% dropout rate, 25 subjects should be enrolled to obtain a final sample size of 20 subjects.

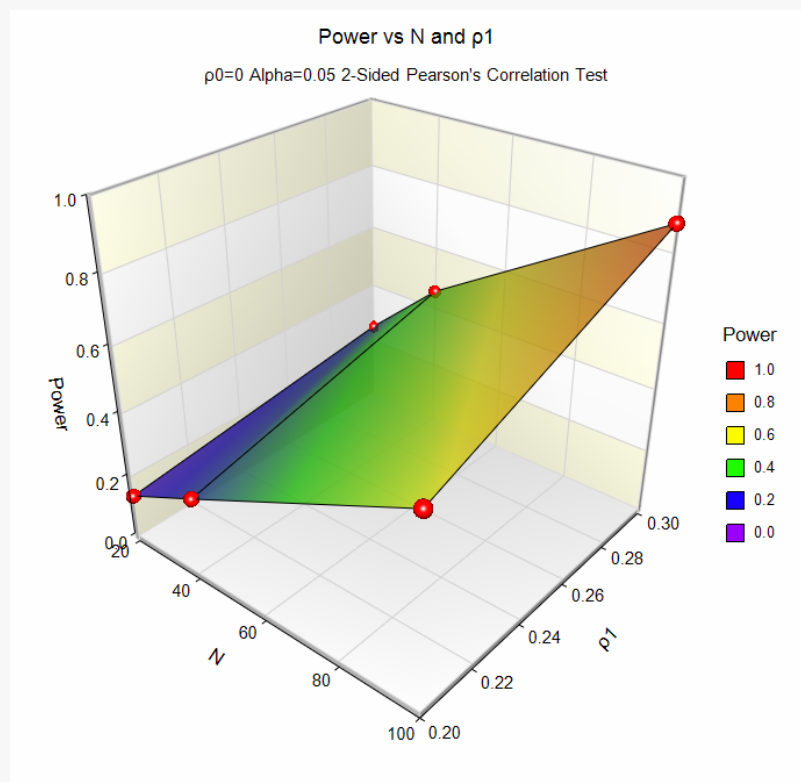
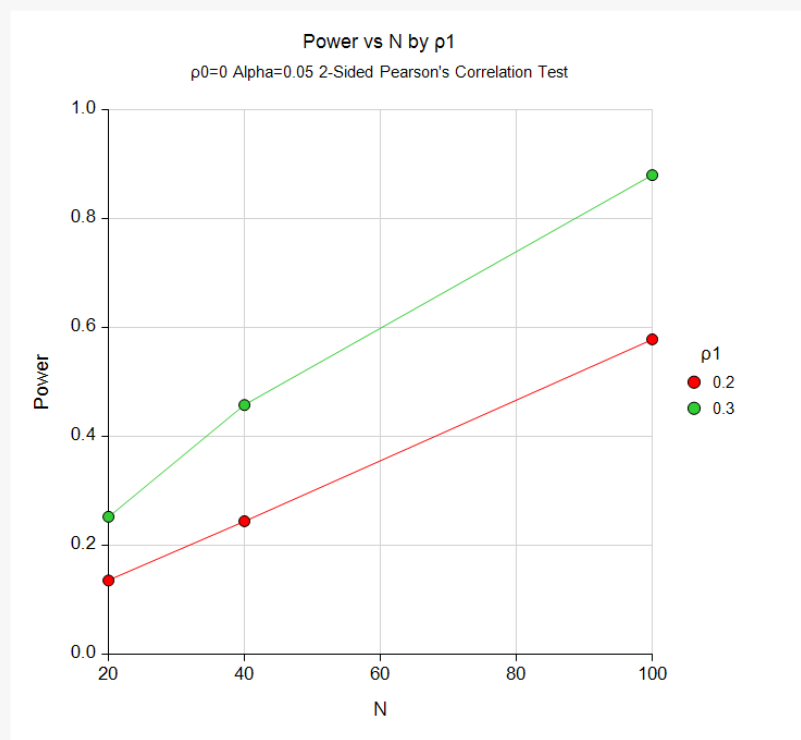
**References**

- Graybill, Franklin. 1961. An Introduction to Linear Statistical Models. McGraw-Hill. New York, New York.
- Guenther, William C. 1977. 'Desk Calculation of Probabilities for the Distribution of the Sample Correlation Coefficient', The American Statistician, Volume 31, Number 1, pages 45-48.
- Zar, Jerrold H. 1984. Biostatistical Analysis. Second Edition. Prentice-Hall. Englewood Cliffs, New Jersey.
- Devroye, Luc. 1986. Non-Uniform Random Variate Generation. Springer-Verlag. New York.

The Numeric Results report gives the basic results. The Power and Alpha Confidence Intervals report provides a confidence interval for each power and alpha value that was simulated. Definitions follow the reports.

## Plots Section

### Plots



This plot shows the relationship between power and sample size in this example.

## Example 2 – Finding the Sample Size

Continuing with the last example, find the sample size necessary to achieve a power of 90% with a 0.05 significance level.

### Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

#### Design Tab

Solve For ..... **Sample Size**  
 Alternative Hypothesis .....  **$\rho_1 \neq \rho_0$**   
 Simulations ..... **5000**  
 Random Seed ..... **55940312** (for Reproducibility)  
 Power ..... **0.90**  
 Alpha ..... **0.05**  
 $\rho_0$  (Correlation|H0) ..... **0.0**  
 $\rho_1$  (Correlation|H1) ..... **0.2 0.3**  
 Distribution ..... **Bivariate Normal**

### Output

Click the Calculate button to perform the calculations and generate the following output.

#### Simulation Summary

Variables	Simulation Distribution	
	Power (H1)	Alpha (H0)
X and Y	Bivariate Normal	Bivariate Normal

Random Number Pool Size 10000  
 Number of Simulations 5000  
 Random Seed 55940312 (User-Entered)  
 Run Time 15.08 seconds

#### Numeric Results

Solve For: [Sample Size](#)  
 Correlation Hypotheses: H0:  $\rho_1 = \rho_0$  vs. H1:  $\rho_1 \neq \rho_0$

Scenario	Sample Size N	Power		Correlation		Alpha	
		Target	Actual	$\rho_0$	$\rho_1$	Target	Actual
1	269	0.9	0.901	0	0.2	0.05	0.049
2	107	0.9	0.900	0	0.3	0.05	0.048

The required sample size is 269 for  $\rho_1 = 0.2$  and 107 for  $\rho_1 = 0.3$ .



## Example 3 – Validation using Zar (1984)

Zar (1984) page 312 presents an example in which the power of a correlation coefficient is calculated. If  $N = 12$ ,  $\alpha = 0.05$ ,  $\rho_0 = 0$ , and  $\rho_1 = 0.866$ , Zar calculates a power of 98% for a two-sided test.

For reproducibility, we'll use a random seed of 6066504.

### Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 3** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

#### Design Tab

Solve For .....	<b>Power</b>
Alternative Hypothesis .....	<b><math>\rho_1 \neq \rho_0</math></b>
Simulations .....	<b>5000</b>
Random Seed .....	<b>6066504</b> (for Reproducibility)
Alpha.....	<b>0.05</b>
N (Sample Size).....	<b>12</b>
$\rho_0$ (Correlation H0) .....	<b>0.0</b>
$\rho_1$ (Correlation H1) .....	<b>0.866</b>
Distribution.....	<b>Bivariate Normal</b>

### Output

Click the Calculate button to perform the calculations and generate the following output.

#### Simulation Summary

Variables	Simulation Distribution	
	Power (H1)	Alpha (H0)
X and Y	Bivariate Normal	Bivariate Normal
Random Number Pool Size	10000	
Number of Simulations	5000	
Random Seed	6066504 (User-Entered)	
Run Time	0.17 seconds	

## Pearson's Correlation Tests (Simulation)

**Numeric Results**

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Solve For: **Power**  
Correlation Hypotheses:  $H_0: \rho_1 = \rho_0$  vs.  $H_1: \rho_1 \neq \rho_0$

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Scenario	Sample Size N	Power	Correlation		Alpha	
			$\rho_0$	$\rho_1$	Target	Actual
1	12	0.981	0	0.866	0.05	0.048

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The power of 0.981 matches Zar's results.

## Example 4 – Validation using Graybill (1961)

Graybill (1961) pages 211-212 presents an example in which the power of a correlation coefficient is calculated when the baseline correlation is different from zero. Let  $N = 24$ ,  $\alpha = 0.05$ , and  $\rho_0 = 0.5$ . Graybill calculates the power of a two-sided test when  $\rho_1 = 0.2$  and  $0.3$  to be  $0.363$  and  $0.193$ .

For reproducibility, we'll use a random seed of 5795811.

### Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 4** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

#### Design Tab

Solve For .....	<b>Power</b>
Alternative Hypothesis .....	<b><math>\rho_1 \neq \rho_0</math></b>
Simulations .....	<b>5000</b>
Random Seed.....	<b>5795811</b> (for Reproducibility)
Alpha.....	<b>0.05</b>
N (Sample Size).....	<b>24</b>
$\rho_0$ (Correlation H0) .....	<b>0.5</b>
$\rho_1$ (Correlation H1) .....	<b>0.2 0.3</b>
Distribution.....	<b>Bivariate Normal</b>

### Output

Click the Calculate button to perform the calculations and generate the following output.

#### Simulation Summary

Variables	Simulation Distribution	
	Power (H1)	Alpha (H0)
X and Y	Bivariate Normal	Bivariate Normal
Random Number Pool Size	10000	
Number of Simulations	5000	
Random Seed	5795811 (User-Entered)	
Run Time	0.55 seconds	

## Pearson's Correlation Tests (Simulation)

**Numeric Results**

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Solve For: **Power**  
Correlation Hypotheses:  $H_0: \rho_1 = \rho_0$  vs.  $H_1: \rho_1 \neq \rho_0$

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Scenario	Sample Size N	Power	Correlation		Alpha	
			$\rho_0$	$\rho_1$	Target	Actual
1	24	0.372	0.5	0.2	0.05	0.056
2	24	0.192	0.5	0.3	0.05	0.051

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The power values match Graybill's results.