Chapter 743

Superiority by a Margin Tests for Simple Linear Regression

Introduction

This procedure computes power and sample size for superiority by a margin tests of the slope in simple linear regression. Simple linear regression is a commonly used procedure in statistical analysis to model a linear relationship between a dependent variable Y and an independent variable X.

The details of sample size calculation for simple linear regression are presented in the *Non-Zero Null Tests for Simple Linear Regression* chapter and they will not be duplicated here. This chapter only discusses those changes necessary for superiority by a margin tests.

Difference between Simple Linear Regression and Correlation

The correlation coefficient is used when *X* and *Y* are from a bivariate normal distribution. That is, *X* is assumed to be a random variable whose distribution is normal. The values of X will not be known until the study is completed. In the simple linear regression context, no statement is made about the distribution of *X*. In fact, *X* does not have to be a random variable. In this procedure the distribution of Y is conditioned on X.

Fixed or Random X

Gatsonis and Sampson (1989) present power analysis results for two approaches: *unconditional* and *conditional*. This procedure provides a calculation for the *conditional* (fixed X) approach.

The *unconditional* approach assumes that X is normally distributed and is based on the correlation coefficient. The normality assumption might occasionally be met, but not frequently. Our impression is that usually the values of X will not be known at the planning stage, and they will not follow (even approximately) the normal distribution. Hence, the only option available is to proceed with the sample size calculation using the *conditional* approach and then estimate the standard deviation of the X's as best you can.

Technical Details

Suppose that the dependence of a variable *Y* on another variable *X* can be modeled using the simple linear equation

$$Y = A + BX$$

In this equation, *A* is the *Y*-intercept, *B* is the slope, *Y* is the dependent variable, and *X* is the independent variable.

The nature of the relationship between *Y* and *X* is studied using a sample of *N* observations. Each observation consists of a data pair: the *X* value and the *Y* value. The values of *A* and *B* are estimated from these observations using the method of least squares. Using these estimated values, each data pair may be modeled using the equation

$$Y_i = a + bX_i + e_i$$

Note that *a* and *b* are the least squares estimates of the population parameters *A* and *B*. The *e* values represent the discrepancies between the estimated values (*a* + *bX*) and the actual values *Y*. They are called the errors or residuals.

If it is assumed that these *e* values are normally distributed, tests of hypotheses about *A* and *B* can be constructed. Specifically, we can employ a T-test to test the superiority of B compared to a superiority boundary *B0*.

Linear Regression Slope T-Test Statistic

It is anticipated that a one-sided *t*-test of a regression coefficient will be used to conduct the test. Hence, the formula of the test statistic is

$$t_{N-2} = \frac{b - B0}{s_b}$$

where *N* is the sample size, *b* is the estimate of *B*, and s_b is the standard error of *b*.

The Statistical Hypotheses

Non-inferiority and superiority tests are examples of directional (one-sided) tests and their power and sample size could be calculated using the *Simple Linear Regression with a Non-Zero Null* procedure. However, at the urging of our users, we have developed this procedure which provides the input and output options that are convenient for superiority by a margin tests. This section will review the specifics of superiority by a margin testing.

Remember that in the usual simple linear regression setting, the null (H0) and alternative (H1) hypotheses for one-sided upper-tail tests are defined as

$$H0: B \le B0 \quad \text{versus} \quad H1: B > B0$$

Rejecting H0 implies that the slope is larger than the value B0. This test is called an *upper-tail test* because H0 is rejected in samples in which the sample slope is larger than B0.

The corresponding *lower-tail test* is

$$H1: B \ge B0 \quad \text{versus} \quad H1: B < B0$$

Superiority by a margin tests are special cases of the above directional tests. It will be convenient to adopt the following specialize notation for the discussion of these tests.

Parameter	Interpretation	
B	*Population slope.*	This parameter will be estimated using the method of least squares.
B1	*Actual population slope at which power is calculated.*	
		This is the assumed population slope used in all calculations.
B0	*Superiority Slope.*	This is the smallest (or largest) value of the slope for which the new treatment will still be considered superior to the current treatment.
B_R	*Reference value.*	Usually, this is the slope in the reference population.
SM	*Margin of superiority.*	
		This is a tolerance value that defines the magnitude of slope that is not of practical importance. This may be thought of as the largest difference from the reference value that is considered to be trivial.

Superiority by a Margin Tests

A *superiority test* tests that the slope is better than that of the baseline (reference) population by more than a small margin. The actual direction of the hypothesis depends on the whether higher values of the response are good or bad.

Case 1: High Values Good

In this case, higher values are better. The hypotheses are arranged so that rejecting the null hypothesis implies that the slope is greater than a small amount above the reference value. Equivalent sets of the null and alternative hypotheses are

$H0: B \leq B0$ versus $H1: B > B0$

$H0: B \leq B_R + SM$ versus $H1: B > B_R + SM$

Case 2: High Values Bad

In this case, lower values are better. The hypotheses are arranged so that rejecting the null hypothesis implies that the mean of the treatment group is no more than a small amount above the reference value. Equivalent sets of the null and alternative hypotheses are

$H0: B \geq B0$ versus $H1: B < B0$

$H0: B \geq B_R - SM$ versus $H1: B < B_R - SM$

Power Calculation of the Superiority Test of the Regression Coefficient, B

The following presentation is based on the standard results for a t-test as shown by Neter, Wasserman, and Kutner (1983) pages 71 and 72.

The power for a superiority test in which higher values of B are better is calculated as follows for a one-tailed test in which the statistical hypotheses are H0: $B \leq B0$ vs. H1: $B > B0$.

1. Find $t_{1-\alpha}$ such that $T_{df}(t_{1-\alpha}) = 1 - \alpha$, where $T_{df}(x)$ is the area under a central-*t* curve to the left of *x* and *df* = N – 2.

2. Calculate: $X_0 = (t_{1-\alpha})\sigma_e/\sqrt{N}$.

3. Calculate the noncentrality parameter: $\lambda = \sqrt{N}(B1 - B0)\sigma_X/\sigma_e$, where σ_X is the standard deviation of the X values in the regression and *B1* is the slope at which the power is to be calculated.

4. Calculate: $t_1 = \left(X_0 - (B1 - B0)\sigma_X\sqrt{N}/\sigma_e\right) + \lambda$.

5. Power = $1 - T'_{df,\lambda}(t_1)$, where $T'_{df,\lambda}(x)$ is the area to the left of *x* under a noncentral-*t* curve with degrees of freedom *df* and noncentrality parameter λ.

The power for the case in which lower values of B are better is calculated using a test in which $H1: B < B0$.

The sample size can be easily found using a binary search with this power formula.

Calculation of σ$_X$

The above calculation requires the value of σ_X, the (population) standard deviation of the X values in the regression analysis. Except for the occasional experimental design that includes a specification of the X values (e.g., doses), the specific X values are unknown in the planning phase. Hence, a reasonable estimate must be found. **PASS** includes a special tool called the *Standard Deviation Estimator* that will aid in your search for accurate estimates of this parameter.

The following table provides examples of typical data configurations and their corresponding standard deviations.

σ_X	X Values	σ_X	X Values	σ_X	X Values	σ_X	X Values
0.500	1, 2	0.816	1, 2, 3	1.118	1, 2, 3, 4	1.414	1, 2, 3, 4, 5
1.000	1, 3	1.633	1, 3, 5	2.236	1, 3, 5, 7	2.828	1, 3, 5, 7, 9
1.500	1, 4	2.449	1, 4, 7	3.354	1, 4, 7, 10	4.243	1, 4, 7, 10, 13
2.000	1, 5	3.266	1, 5, 9	4.472	1, 5, 9, 13	5.657	1, 5, 9, 13, 17
4.000	1, 9	6.532	1, 9, 17	8.944	1, 9, 17, 25	11.314	1, 9, 17, 25, 33

Because of the direct impact on the power and sample size, it will be important to spend some time determining appropriate values for this parameter.

One final note: when a basic pattern is repeated, its population standard deviation remains the same. For example, the standard deviation of the values 1, 2, 1, 2, 1, 2, 1, 2 is 0.5. This is also the standard deviation of 1, 2 or 1, 2, 1, 2.

Example 1 – Calculating the Power

Suppose a power analysis is required for a simple linear regression study that will perform a superiority test the relationship between two variables, *Y* and *X*. Further suppose that in the past, the slope has been 1 and it will be advantageous to show that it is now at least 1.2.

The analysis will look at the power of several sample sizes between 50 and 250. A superiority test will be used with a significance level of 0.025. Based on previous studies, σe will be assumed to be 0.6. The value of σx will assume that X is binary with equally-likely values of 1 and 2. The power will be computed at B1 = 0.9, 1, 1.1, and 1.2.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For ....**Power**
Higher Slopes Are..**Better (H1: B > B0)**
Alpha..**0.025**
N (Sample Size)..**50 100 150 200 250**
B0 (Superiority Slope)...................................**1.2**
B1 (Slope|H1) **1.3 1.4 1.5**
σx Input Type...**List of X Values**
List of X Values...**1 2**
σe Input Type...**σe (Std Dev of Residuals)**
σe (Std Dev of Residuals)..............................**0.6**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Reports

Numeric Results

Solve For: Power
Hypotheses: H0: B ≤ B0   vs.   H1: B > B0

Power	Sample Size N	Slope		Standard Deviation			R-Squared	
		Superiority B0	H1 (Actual) B1	X σx	Y σY	Residuals σe	R²	Alpha
0.0834	50	1.2	1.3	0.5	0.885	0.6	0.540	0.025
0.2104	50	1.2	1.4	0.5	0.922	0.6	0.576	0.025
0.4100	50	1.2	1.5	0.5	0.960	0.6	0.610	0.025
0.1282	100	1.2	1.3	0.5	0.885	0.6	0.540	0.025
0.3784	100	1.2	1.4	0.5	0.922	0.6	0.576	0.025
0.6969	100	1.2	1.5	0.5	0.960	0.6	0.610	0.025
0.1721	150	1.2	1.3	0.5	0.885	0.6	0.540	0.025
0.5271	150	1.2	1.4	0.5	0.922	0.6	0.576	0.025
0.8604	150	1.2	1.5	0.5	0.960	0.6	0.610	0.025
0.2156	200	1.2	1.3	0.5	0.885	0.6	0.540	0.025
0.6501	200	1.2	1.4	0.5	0.922	0.6	0.576	0.025
0.9404	200	1.2	1.5	0.5	0.960	0.6	0.610	0.025
0.2587	250	1.2	1.3	0.5	0.885	0.6	0.540	0.025
0.7470	250	1.2	1.4	0.5	0.922	0.6	0.576	0.025
0.9760	250	1.2	1.5	0.5	0.960	0.6	0.610	0.025

Power   The probability of rejecting a false null hypothesis when the alternative hypothesis is true.
N   The size of the sample drawn from the population.
B0   The superiority slope (boundary) in the case in which HIGHER slope values are desired.
B1   The slope under the alternative hypothesis, H1. This is the slope at which the power is calculated.
σx   The standard deviation of the X values.
σY   The standard deviation of Y (ignoring X).
σe   The standard deviation of the residuals.
R²   The R-squared value when Y is regressed on X.
Alpha   The probability of rejecting a true null hypothesis.

Summary Statements

A simple linear regression (single group, Y versus X) design (where higher slopes are considered better) will be used to test whether the slope (B) is superior by a margin, with a superiority slope of 1.2 (H0: B ≤ 1.2 versus H1: B > 1.2). The comparison will be made using a one-sided simple linear regression slope t-test with a Type I error rate (α) of 0.025. The standard deviation of X is assumed to be 0.5 (calculated from the given list of X values), and the standard deviation of residuals is assumed to be 0.6 (corresponding to a standard deviation of Y of 0.885 and an R² of 0.54). To detect a slope of 1.3 with a sample size of 50, the power is 0.0834.

Dropout-Inflated Sample Size

Dropout Rate	Sample Size N	Dropout-Inflated Enrollment Sample Size N'	Expected Number of Dropouts D
20%	50	63	13
20%	100	125	25
20%	150	188	38
20%	200	250	50
20%	250	313	63

Dropout Rate   The percentage of subjects (or items) that are expected to be lost at random during the course of the study and for whom no response data will be collected (i.e., will be treated as "missing"). Abbreviated as DR.

N   The evaluable sample size at which power is computed (as entered by the user). If N subjects are evaluated out of the N' subjects that are enrolled in the study, the design will achieve the stated power.

N'   The total number of subjects that should be enrolled in the study in order to obtain N evaluable subjects, based on the assumed dropout rate. N' is calculated by inflating N using the formula N' = N / (1 - DR), with N' always rounded up. (See Julious, S.A. (2010) pages 52-53, or Chow, S.C., Shao, J., Wang, H., and Lokhnygina, Y. (2018) pages 32-33.)

D   The expected number of dropouts. D = N' - N.

Dropout Summary Statements

Anticipating a 20% dropout rate, 63 subjects should be enrolled to obtain a final sample size of 50 subjects.
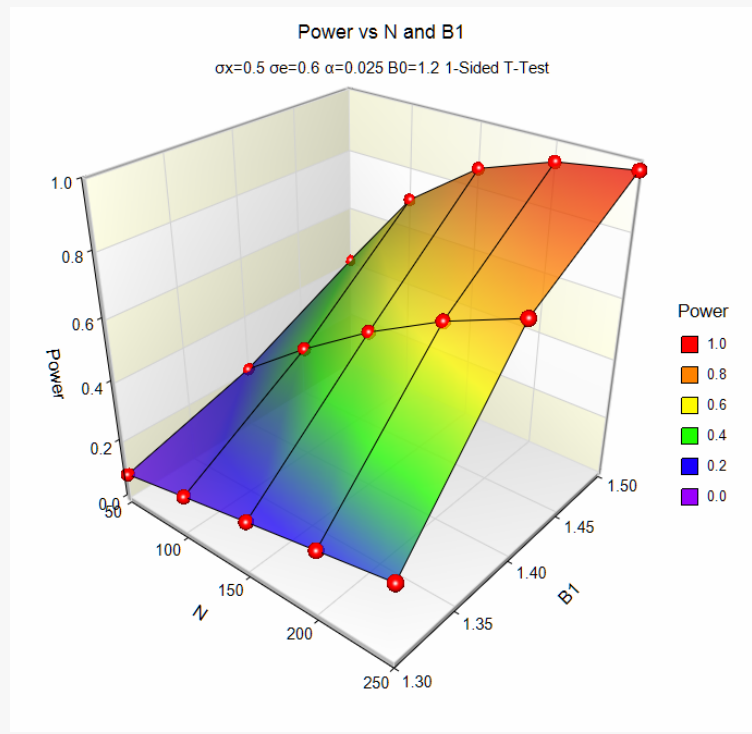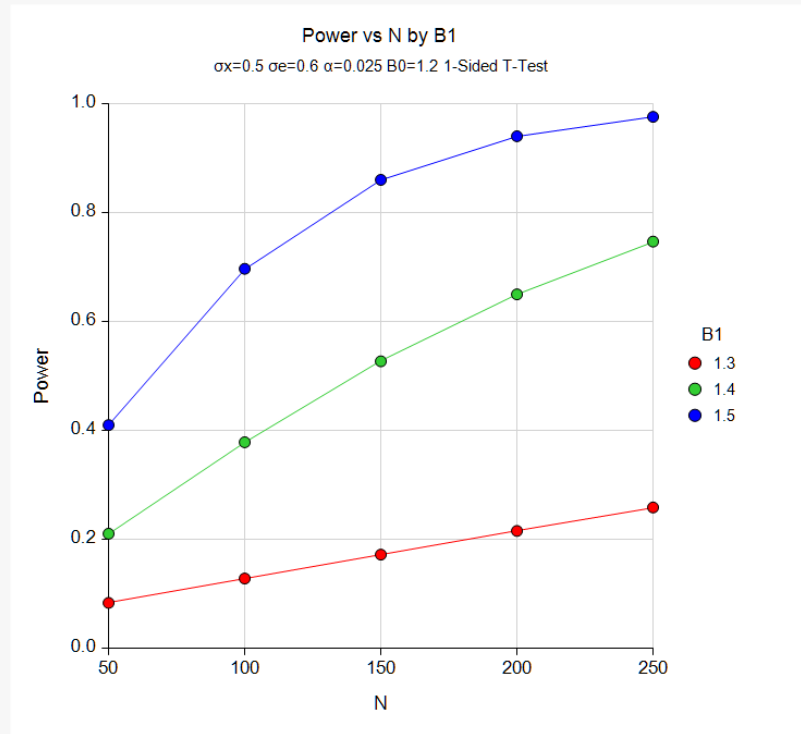

References

Dupont, W.D. and Plummer, W.D. Jr. 1998. 'Power and Sample Size Calculations for Studies Involving Linear Regression'. Controlled Clinical Trials, Vol. 19, Pages 589-601.

Sampson, Allan R. 1974. 'A Tale of Two Regressions'. JASA, Vol. 69, No. 347, Pages 682-689.

Neter, J., Wasserman, W., and Kutner, M. 1983. Applied Linear Regression Models. Richard D. Irwin, Inc. Chicago, Illinois.

This report shows the calculated power for each of the scenarios.

Plots Section

Plots
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These plots show the power versus the sample size for the three values of B1.

Example 2 – Validation using Another PASS Procedure

We could not find a validation example for this procedure in the literature. But since this procedure was derived from the **Non-Zero Null Tests for Simple Linear Regression** procedure, we will use an example from that previously validated procedure to validate this procedure.

In the other procedure, set the *Solve For* parameter to Sample Size. Also, set the *Alternative Hypothesis* to One-Sided (H1: B > B0). Also, set $Power = 0.9, B0 = 1.2, B1 = 1.4, \alpha = 0.05, \sigma_X = 0.5, \sigma_e = 0.6$.

The sample size is calculated as 310.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For ....**Sample Size**
Higher Slopes Are...**Better (H1: B > B0)**
Power..**0.90**
Alpha..**0.05**
B0 (Superiority Slope)....................................**1.2**
B1 (Slope|H1) **1.4**
σx Input Type...**σx (Std Dev of X)**
σx (Std Dev of X) **0.5**
σe Input Type...**σe (Std Dev of Residuals)**
σe (Std Dev of Residuals)..............................**0.6**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Solve For: Sample Size
Hypotheses: H0: B ≤ B0   vs.   H1: B > B0

		Slope		Standard Deviation				
Power	Sample Size N	Superiority B0	H1 (Actual) B1	X σx	Y σY	Residuals σe	R-Squared R²	Alpha
0.9003	310	1.2	1.4	0.5	0.922	0.6	0.576	0.05

The sample size of 310 matches the result from the other procedure.