

## Chapter 246

# Superiority by a Margin Tests for the Difference Between Two Poisson Rates in a Cluster-Randomized Design

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## Introduction

This procedure calculates power and sample size for superiority by a margin tests of two rates in a cluster-randomized design in which the outcome variable is a count. It uses the work of Wang, Zhang, and Ahn (2018) which give the power for the case of varying cluster sizes. The analysis uses a simple z-test comparing the two rates.

Cluster-randomized designs are those in which whole clusters of subjects (classes, hospitals, communities, etc.) are put into the treatment group or the control group. Generally speaking, the larger the cluster sizes and the higher the correlation among subjects within the same cluster, the larger will be the overall sample size necessary to detect an effect with the same power.

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## The Statistical Hypotheses

Superiority by a margin tests are examples of directional (one-sided) tests. This program module provides the input and output in formats that are convenient for these types of tests. This section will review the specifics of superiority by a margin testing.

Remember that in the usual test setting, the null ( $H_0$ ) and alternative ( $H_1$ ) hypotheses for one-sided tests are defined as follows, assuming that  $\delta = \mu_1 - \mu_2$  is to be tested.

$$H_0: \delta \leq 0 \quad \text{versus} \quad H_1: \delta > 0$$

Rejecting this test implies that the mean difference is larger than the value  $\delta$ . This test is called an *upper-tailed test* because it is rejected in samples in which the difference between the sample means is larger than  $D$ .

Following is an example of a *lower-tailed test*.

$$H_0: \delta \geq 0 \quad \text{versus} \quad H_1: \delta < 0$$

Superiority by a margin tests are special cases of the above directional tests. It will be convenient to adopt the following specialized notation for the discussion of these tests.

## Superiority by a Margin Tests for the Difference of Two Poisson Rates in a Cluster-Randomized Design

<b>Parameter</b>	<b>PASS Input/Output</b>	<b>Interpretation</b>
$\lambda_1$	$\lambda_1$	<i>Incidence rate</i> of population 1. Population 1 is assumed to consist of those who have received the new treatment.
$\lambda_2$	$\lambda_2$	<i>Incidence rate</i> of population 2. Population 2 is assumed to consist of those who have received the control (standard or reference) treatment.
$\varepsilon$	D0	<i>Margin of Superiority.</i> This is a tolerance value that defines the magnitude of the amount that is not of practical importance. This may be thought of as the largest change from the baseline that is considered to be trivial. The absolute value is shown to emphasize that this is a magnitude. The sign of the value will be determined by the specific design that is being used.
$\delta$	D1	<i>True difference.</i> This is the value of $\lambda_1 - \lambda_2$ , the difference between the rates.

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## Superiority by a Margin Tests

A *superiority by a margin test* tests that the treatment rate is better than the control rate by more than the superiority margin. The actual direction of the hypothesis depends on the nature of the response variable being studied.

### Case 1: High Values are Better

In this case, higher values are better. The hypotheses are arranged so that rejecting the null hypothesis implies that the treatment rate is no less than some amount above the control rate. The following are equivalent sets of hypotheses.

$$H_0: \delta \leq \varepsilon \quad \text{versus} \quad H_1: \delta > \varepsilon, \quad \varepsilon > 0$$

$$H_0: \lambda_1 - \lambda_2 \leq \varepsilon \quad \text{versus} \quad H_1: \lambda_1 - \lambda_2 > \varepsilon, \quad \varepsilon > 0$$

$$H_0: \lambda_1 \leq \lambda_2 + \varepsilon \quad \text{versus} \quad H_1: \lambda_1 > \lambda_2 + \varepsilon, \quad \varepsilon > 0$$

### Case 2: High Values are Worse

In this case, lower values are better. The hypotheses are arranged so that rejecting the null hypothesis implies that the treatment rate is no more than some amount below the control rate. The following are equivalent sets of hypotheses.

$$H_0: \delta \geq -\varepsilon \quad \text{versus} \quad H_1: \delta < -\varepsilon, \quad \varepsilon > 0$$

$$H_0: \lambda_1 - \lambda_2 \geq -\varepsilon \quad \text{versus} \quad H_1: \lambda_1 - \lambda_2 < -\varepsilon, \quad \varepsilon > 0$$

$$H_0: \lambda_1 \geq \lambda_2 - \varepsilon \quad \text{versus} \quad H_1: \lambda_1 < \lambda_2 - \varepsilon, \quad \varepsilon > 0$$

## Technical Details

The following discussion summarizes the results in Wang, Zhang, and Ahn (2018).

Suppose you are interested in comparing the incidence rates of two groups (treatment and control) with a superiority by a margin test. Further suppose that the response is known to be related to other covariates (such as age, race, or gender) and so their impact needs to be adjusted for. This may be accomplished by stratifying on the covariates and forming hypotheses about a common mean difference across all clusters and strata. Often, the stratification is based on cluster size, but this is not required.

Let  $Y_{jki}$  be the count outcome of the  $i$ th ( $i = 1, \dots, M_{jk}$ ) subject in the  $k$ th ( $k = 1, \dots, K_j$ ) cluster of the  $j$ th ( $j = 1, 2$ ) group. Assuming that  $Y_{jki}$  follows a Poisson model with

$$\lambda_j = E(Y_{jki}) = \text{Var}(Y_{jki})$$

and a common intracluster correlation coefficient (ICC)  $\rho = \text{corr}(Y_{jki}, Y_{jki'})$  for  $i \neq i'$ .

## Test Statistic

An unbiased estimator of  $\lambda_j$  is

$$\hat{\lambda}_j = \frac{\sum_{k=1}^{K_j} \sum_{i=1}^{M_{jk}} Y_{jki}}{\sum_{k=1}^{K_j} M_{jk}}$$

with estimated variance

$$s_j^2 = \frac{\hat{\lambda}_j \sum_{k=1}^{K_j} M_{jk} [1 + (M_{jk} - 1)\hat{\rho}]}{\left(\sum_{k=1}^{K_j} M_{jk}\right)^2}$$

Using a one-sided hypothesis  $H_0: \lambda_1 = \lambda_2$  versus  $H_1: \lambda_1 < \lambda_2$ ,  $H_0$  is rejected if

$$\frac{\hat{\lambda}_1 - \hat{\lambda}_2}{\sqrt{s_1^2 + s_2^2}} > z_{1-\alpha}$$

## Sample Size and Power

Wang, Zhang, and Ahn (2018) provide the following formula for estimating  $K_2$  (the number of clusters in the control group) when there are an unequal number of clusters in each group.

$$K_2 = \frac{\left(z_{1-\frac{\alpha}{2}} + z_{1-\beta}\right)^2 \left(\frac{\lambda_1}{R} + \lambda_2\right)}{(\lambda_1 - \lambda_2)^2} \left[ (1 - \rho) \frac{1}{\theta} + \rho + \rho\gamma^2 \right]$$

where  $R = K_1/K_2$ ,  $M = E(M_{jk})$ ,  $\tau^2 = \text{Var}(M_{jk})$ , and  $\gamma = \tau/M$ . Here  $M$  is the average cluster size of all clusters in the study and  $\gamma$  is the coefficient of variation of the cluster sizes. Note that  $z_x = \Phi(x)$  is the standard normal distribution function.

This equation can easily be rearranged to provide a formula for power  $(1 - \beta)$ .

## Example 1 – Finding the Number of Clusters

A superiority study is being planned to investigate whether a new intervention will decrease the incidence rate of a certain disease over the rate achieved by the current intervention. This response is a count. For a number of reasons, the researchers decide to administer the intervention to whole clusters (clinics) rather than randomize the treatment to individuals within a cluster. The number of clinics receiving each treatment will be balanced.

The average number of subjects per clinic is 21. The coefficient of variation of the cluster sizes is 0.42.

Prior studies have obtained an incidence rate of 0.35 for the current treatment and an ICC of 0.07. The superiority margin is set to 0.05. The researchers want to compare the necessary sample size when the actual incidence rate of the new treatment is 0.3, 0.25, 0.2, and 0.15.

The one-sided significance level is set to 0.025 and the power is set to 0.8.

### Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

#### Design Tab

Solve For .....	<b>K1 (Number of Clusters in Group 1)</b>
Higher Rates Are .....	<b>Worse (H1: <math>\lambda_1 - \lambda_2 &lt; D_0</math>, where <math>D_0 &lt; 0</math>)</b>
Power.....	<b>0.80</b>
Alpha.....	<b>0.025</b>
K2 (Clusters in Group 2) .....	<b>K1</b>
M (Average of Cluster Sizes) .....	<b>21</b>
CV (Coef of Variation of Cluster Sizes) .....	<b>0.42</b>
Incidence Rate Input Type .....	<b>Differences (D0 and D1)</b>
D0 (Superiority Difference) .....	<b>-0.05</b>
D1 (Actual Difference) .....	<b>-0.2 -0.15 -0.1</b>
$\lambda_2$ (Incidence Rate of Control Group) .....	<b>0.35</b>
$\rho$ (Intraclass Correlation, ICC) .....	<b>0.07</b>

## Output

Click the Calculate button to perform the calculations and generate the following output.

## Numeric Reports

### Numeric Results

Solve For: K1 (Number of Clusters in Group 1)  
 Groups: 1 = Treatment, 2 = Control  
 Higher Rates Are: Worse  
 Hypotheses:  $H_0: \lambda_1 - \lambda_2 \geq D_0$  vs.  $H_1: \lambda_1 - \lambda_2 < D_0$

Power	Number of Clusters			Cluster Size		Total Sample Size N	Incidence Rate					Intraclass Correlation $\rho$	Alpha
	Treatment K1	Control K2	Total K	Average M	CV		Treatment		Control $\lambda_2$	Difference			
							Superiority $\lambda_{1.0}$	Actual $\lambda_{1.1}$		Superiority D0	Actual D1		
0.80020	179	179	358	21	0.42	7518	0.3	0.15	0.35	-0.05	-0.20	0.07	0.025
0.80042	199	199	398	21	0.42	8358	0.3	0.20	0.35	-0.05	-0.15	0.07	0.025
0.80060	219	219	438	21	0.42	9198	0.3	0.25	0.35	-0.05	-0.10	0.07	0.025

Power The probability of rejecting a false null hypothesis when the alternative hypothesis is true.  
 K1, K2, and K The number of clusters in groups 1, 2, and both, respectively.  
 M The average cluster size. It is the anticipated average number of subjects per cluster.  
 CV The coefficient of variation of the cluster sizes. It is the standard deviation of cluster sizes divided by the average cluster size.  
 N The total sample size, i.e., the total number of subjects from all clusters.  
 $\lambda_{1.0}$  The superiority bound on incidence rate in the treatment group.  
 $\lambda_{1.1}$  The actual incidence rate of the treatment group at which the power is computed.  
 $\lambda_2$  The incidence rate in the control group.  
 D0 The difference in incidence rates assumed by the null hypothesis. This is often called the superiority margin.  
 $D_0 = \lambda_{1.0} - \lambda_2$ .  
 D1 The difference in incidence rates assumed by the alternative hypothesis. This is the difference at which the power is computed.  $D_1 = \lambda_{1.1} - \lambda_2$ .  
 $\rho$  The intraclass correlation coefficient. This is the correlation between any two subjects within a particular cluster.  
 Alpha The probability of rejecting a true null hypothesis.

### Summary Statements

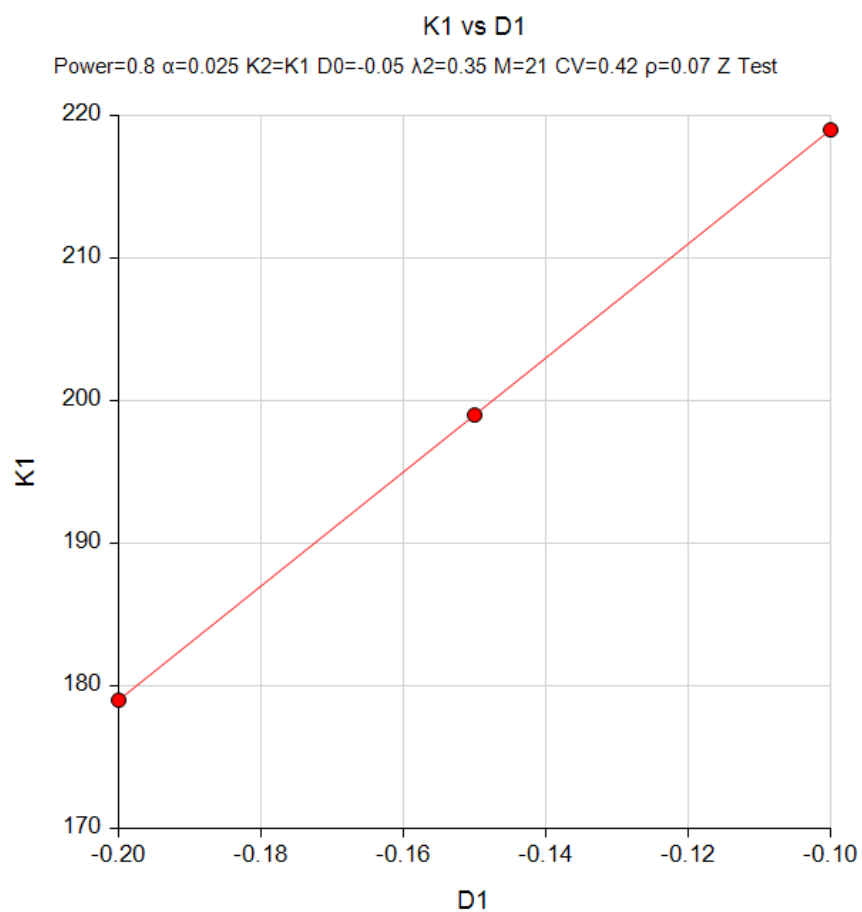
A parallel two-group cluster-randomized design will be used to test whether the Group 1 (treatment) incidence rate ( $\lambda_1$ ) is superior to the Group 2 (control) incidence rate ( $\lambda_2$ ) by a margin, with an incidence rate difference superiority margin of -0.05 ( $H_0: \delta \geq -0.05$  versus  $H_1: \delta < -0.05$ ,  $\delta = \lambda_1 - \lambda_2$ ). The comparison will be made using a one-sided incidence rate difference Z-test with a Type I error rate ( $\alpha$ ) of 0.025. The intraclass correlation coefficient is assumed to be 0.07. The control group incidence rate is assumed to be 0.35. The individual cluster sizes (the number of subjects per cluster) are assumed to vary according to a discrete distribution with mean 21 and coefficient of variation 0.42. To detect an incidence rate difference ( $\lambda_1 - \lambda_2$ ) of -0.2 (or treatment group incidence rate of 0.15) with 80% power, with an average of 21 subjects per cluster, the number of clusters needed will be 179 in Group 1 (treatment) and 179 in Group 2 (control) (for an overall total of 7518 subjects).

### References

Wang, J., Zhang, S., and Ahn, C. 2018. 'Sample size calculation for count outcomes in cluster randomization trials with varying cluster sizes.' Communications in Statistics--Theory and Methods, DOI: 10.1080/03610926.2018.1532004.

## Plots Section

### Plots



The values from the Numeric Results report are displayed in this plot.

## Example 2 – Validation using a Previously Validated Procedure

We could not find a validation example in the literature so we will use a previously validated procedure in **PASS** to validate this procedure.

Suppose in a superiority design, higher rates are better, power = 0.9, alpha = 0.025,  $K_2 = K_1$ ,  $M = 50$ ,  $CV = 0.2$ ,  $\lambda_{1.0} = 0.6$ ,  $\lambda_{1.1} = 0.5$ ,  $\lambda_2 = 0.5$ , and  $\rho = 0.002$ . Solve for  $K_1$ .

This scenario can be solved using the procedure “Tests for the Difference Between Two Poisson Rates in a Cluster-Randomized Design” with the following settings.

Set the alternative hypothesis to “One-Sided ( $H_1: \delta > 0$ )”, power = 0.9, alpha = 0.025,  $K_2 = K_1$ ,  $M = 50$ ,  $CV = 0.2$ ,  $\lambda_1 = 0.6$ ,  $\lambda_2 = 0.5$ , and  $\rho = 0.002$ . The solution is  $K_1 = K_2 = 26$ .

### Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

#### Design Tab

Solve For .....	<b>K1 (Number of Clusters in Group 1)</b>
Higher Rates Are .....	<b>Better (<math>H_1: \lambda_1 - \lambda_2 &gt; D_0</math>, where <math>D_0 &gt; 0</math>)</b>
Power.....	<b>0.9</b>
Alpha.....	<b>0.025</b>
$K_2$ (Clusters in Group 2) .....	<b>K1</b>
$M$ (Average of Cluster Sizes) .....	<b>50</b>
$CV$ (Coef of Variation of Cluster Sizes) .....	<b>0.2</b>
Incidence Rate Input Type .....	<b>Incidence Rates (<math>\lambda_{1.0}</math> and <math>\lambda_{1.1}</math>)</b>
$\lambda_{1.0}$ (Superiority Incidence Rate) .....	<b>0.6</b>
$\lambda_{1.1}$ (Actual Incidence Rate) .....	<b>0.5</b>
$\lambda_2$ (Incidence Rate of Control Group) .....	<b>0.5</b>
$\rho$ (Intraclass Correlation, ICC) .....	<b>0.002</b>

## Superiority by a Margin Tests for the Difference of Two Poisson Rates in a Cluster-Randomized Design

## Output

Click the Calculate button to perform the calculations and generate the following output.

### Numeric Results

Solve For: [K1 \(Number of Clusters in Group 1\)](#)  
 Groups: 1 = Treatment, 2 = Control  
 Higher Rates Are: Better  
 Hypotheses:  $H_0: \lambda_1 - \lambda_2 \leq D_0$  vs.  $H_1: \lambda_1 - \lambda_2 > D_0$

Power	Number of Clusters			Cluster Size		Total Sample Size N	Incidence Rate						Intraclass Correlation $\rho$	Alpha
	Treatment K1	Control K2	Total K	Average M	CV		Treatment		Control $\lambda_2$	Difference				
							Superiority $\lambda_{1.0}$	Actual $\lambda_{1.1}$		Superiority D0	Actual D1			
0.90572	26	26	52	50	0.2	2600	0.6	0.5	0.5	0.1	0	0.002	0.025	

**PASS** has also obtained  $K_1 = K_2 = 26$ . Thus, the procedure is validated.