

## Chapter 514

# Superiority by a Margin Tests for the Ratio of Two Poisson Rates in a 2x2 Cross-Over Design

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## Introduction

Senn (2002) defines a *cross-over* design as one in which each subject receives all treatments, and the objective is to study differences among the treatments. The name *cross-over* comes from the most common case in which there are only two treatments. In this case, each subject *crosses over* from one treatment to the other. It is assumed that there is a *washout* period between treatments during which the response returns back to its baseline value. If this does not occur, there is said to be a *carry-over* effect.

A 2x2 cross-over design contains two *sequences* (treatment orderings) and two time periods (occasions). One sequence receives treatment A followed by treatment B. The other sequence receives B and then A. The design includes a washout period between responses to make certain that the effects of the first drug do not carry over to the second. Thus, the groups in this design are defined by the sequence in which the drugs are administered, not by the treatments they receive. Indeed, higher-order cross-over designs have been used in which the same treatment is used at both occasions.

Cross-over designs are employed because, if the no-carryover assumption is met, treatment differences are measured within a subject rather than between subjects—making a more precise measurement. Examples of the situations that might use a cross-over design are the comparison of anti-inflammatory drugs in arthritis and the comparison of hypotensive agents in essential hypertension. In both cases, symptoms are expected to return to their usual baseline level shortly after the treatment is stopped.

The sample size calculations in the procedure are based on the formulas presented in Lui (2016).

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## Advantages of Cross-Over Designs

A comparison of treatments on the same subject is expected to be more precise. The increased precision often translates into a smaller sample size. Also, patient enrollment into the study may be easier because each patient will receive both treatments. Finally, it is often more difficult to obtain a subject than to obtain a measurement.

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## Disadvantages of Cross-Over Designs

The statistical analysis of a cross-over experiment is more complex than a parallel-group experiment and requires additional assumptions. It may be difficult to separate the treatment effect from the period effect, the carry-over effect of the previous treatment, and the interaction between period and treatment.

The design cannot be used when the treatment (or the measurement of the response) alters the subject permanently. Hence, it should not be used to compare treatments that are intended to provide a cure.

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Because subjects must be measured at least twice, it is often more difficult to keep patients enrolled in the study. It is arguably simpler to measure a subject once than to obtain their measurement twice. This is particularly true when the measurement process is painful, uncomfortable, embarrassing, or time consuming.

## Technical Details

The 2x2 crossover design may be described as follows. Randomly assign the subjects to one of two sequence groups so that there are  $n_1$  subjects in sequence one and  $n_2$  subjects in sequence two. In order to achieve design balance, the sample sizes  $n_1$  and  $n_2$  are assumed to be equal so that  $n_1 = n_2 = n = N/2$ .

Sequence one is given the control (A) followed by the treatment (B). Sequence two is given the treatment (B) followed by the control (A).

## Cross-Over Design

The discussions that follow summarize the results in Lui (2016) on pages 75-88. Consider a 2x2 cross-over design and let  $Y_{ij}^{(g)}$  represent the frequency of event occurrences for the  $j^{\text{th}}$  subject,  $j = 1, \dots, n_g$ , in the  $i^{\text{th}}$  period ( $i = 1, 2$ ), in sequence  $g$  ( $g = 1, 2$ ). Let  $X_{ij}^{(g)}$  represent the treatment-received covariate for the  $j^{\text{th}}$  subject,  $j = 1, \dots, n_g$ , in the  $i^{\text{th}}$  period ( $i = 1, 2$ ), in sequence  $g$  ( $g = 1, 2$ ) such that  $X_{ij}^{(g)} = 1$  for a subject receiving the experimental treatment and  $X_{ij}^{(g)} = 0$  for a subject receiving the control or standard treatment. Let  $Z_{ij}^{(g)}$  represent the period covariate for the  $j^{\text{th}}$  subject,  $j = 1, \dots, n_g$ , in the  $i^{\text{th}}$  period ( $i = 1, 2$ ), in sequence  $g$  ( $g = 1, 2$ ) such that  $Z_{ij}^{(g)} = 1$  for period 2 and  $Z_{ij}^{(g)} = 0$  for period 1. Finally, assume that the  $Y_{ij}^{(g)}$  follow a Poisson distribution with mean

$$E(Y_{ij}^{(g)}) = \mu_j^{(g)} \exp(\eta X_{ij}^{(g)} + \gamma Z_{ij}^{(g)})$$

where  $\mu_j^{(g)}$  represents the random effect of the  $j^{\text{th}}$  subject assigned to sequence  $g$  and has overall mean  $\mu$ ,  $\eta$  is the relative effect of the treatment to the control, and  $\gamma$  is the relative effect of period 2 to period 1. For a fixed period, the ratio,  $R$ , of mean event rates for the treatment versus the control is

$$R = \frac{\lambda_T}{\lambda_C} = e^\eta.$$

Similarly, the ratio of mean event rates for period 2 versus period 1 is

$$R_p = \frac{\lambda_2}{\lambda_1} = e^\gamma.$$

## Superiority by a Margin Test Statistics

### Higher Event Rates Better

When higher event rates are better, the null and alternative hypotheses for a one-sided superiority test are

$$H_0: R \leq R_0 \text{ vs. } H_A: R > R_0$$

or equivalently,

$$H_0: \eta \leq \eta_0 \text{ vs. } H_A: \eta > \eta_0$$

since  $\eta = \log(R)$ .  $R_0$  is the superiority bound (i.e., the smallest event rate ratio ( $\lambda_T/\lambda_C$ ) for which the treatment will be considered superior to the standard or control). When higher event rates are better,  $R_0$  should be greater than one.

The power and sample size calculations are based on the test statistic

$$Z = \frac{\log(\hat{R}) - \log(R_0)}{\sqrt{\widehat{Var}(\log(\hat{R}))}}$$

which is asymptotically distributed as standard normal under the null hypothesis. The event rate ratio estimate,  $\hat{R}$ , and the variance estimate,  $\widehat{Var}(\log(\hat{R}))$ , are calculated as described in Lui (2016) on pages 77 through 79.

The null hypothesis is rejected in favor of the alternative at level  $\alpha$  if

$$Z > Z_{1-\alpha}$$

where  $Z_{1-\alpha}$  is the upper  $1 - \alpha$  percentile of the standard normal distribution.

### Higher Event Rates Worse

When higher event rates are worse, the null and alternative hypotheses for a one-sided superiority test are

$$H_0: R \geq R_0 \text{ vs. } H_A: R < R_0$$

or equivalently,

$$H_0: \eta \geq \eta_0 \text{ vs. } H_A: \eta < \eta_0$$

since  $\eta = \log(R)$ .  $R_0$  is the superiority bound (i.e., the largest event rate ratio ( $\lambda_T/\lambda_C$ ) for which the treatment will be considered superior to the standard or control). When higher event rates are worse,  $R_0$  should be less than one.

The power and sample size calculations are based on the test statistic

$$Z = \frac{\log(\hat{R}) - \log(R_0)}{\sqrt{\widehat{Var}(\log(\hat{R}))}}$$

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which is asymptotically distributed as standard normal under the null hypothesis. The event rate ratio estimate,  $\hat{R}$ , and the variance estimate,  $\widehat{Var}(\log(\hat{R}))$ , are calculated as described in Lui (2016) on pages 77 through 79.

The null hypothesis is rejected in favor of the alternative at level  $\alpha$  if

$$Z < Z_{\alpha}$$

where  $Z_{\alpha}$  is the lower  $\alpha$  percentile of the standard normal distribution.

## Power Calculation

If  $\hat{R}$  is the estimate of the event rate ratio, then  $\hat{\eta} = \log(\hat{R})$  has asymptotic variance

$$\widehat{Var}(\log(\hat{R})) = \frac{V(\mu, \eta, \gamma)}{n}$$

where

$$V(\mu, \eta, \gamma) = \frac{1}{4} \left( \frac{1}{\mu(1 + e^{\eta+\gamma})p_1(1 - p_1)} + \frac{1}{\mu(e^{\eta} + e^{\gamma})p_2(1 - p_2)} \right)$$

with

$$p_1 = \frac{e^{\eta+\gamma}}{1 + e^{\eta+\gamma}}$$

$$p_2 = \frac{e^{\gamma}}{e^{\eta} + e^{\gamma}}.$$

## Higher Event Rates Better

Derived from the sample size formula given in Lui (2016) on page 87, the power for the one-sided superiority test of  $H_0: R \leq R_0$  versus  $H_A: R > R_0$  is

$$\Phi \left( \frac{\sqrt{n}(\log(R_1) - \log(R_0))}{\sqrt{V(\mu, \eta, \gamma)}} - Z_{1-\alpha} \right)$$

where  $\Phi()$  is the standard normal distribution function,  $R_1$  is the actual value of the event rate ratio under the alternative hypothesis, and  $Z_{1-\alpha}$  is the upper  $1 - \alpha$  percentile of the standard normal distribution. The sample size calculation formula is

$$n = \text{Ceiling} \left\{ \left( \frac{(Z_{1-\alpha} + Z_{1-\beta})\sqrt{V(\mu, \eta, \gamma)}}{\log(R_1) - \log(R_0)} \right)^2 \right\}.$$

## Higher Event Rates Worse

Derived from the sample size formula given in Lui (2016) on page 87, the power for the one-sided superiority test of  $H_0: R \geq R_0$  versus  $H_A: R < R_0$  is

$$\Phi\left(\frac{\sqrt{n}(\log(R_0) - \log(R_1))}{\sqrt{V(\mu, \eta, \gamma)}} + Z_\alpha\right)$$

where  $\Phi()$  is the standard normal distribution function,  $R_1$  is the actual value of the event rate ratio under the alternative hypothesis, and  $Z_\alpha$  is the lower  $\alpha$  percentile of the standard normal distribution. The sample size calculation formula is

$$n = \text{Ceiling}\left\{\left(\frac{(Z_{1-\alpha} + Z_{1-\beta})\sqrt{V(\mu, \eta, \gamma)}}{\log(R_1) - \log(R_0)}\right)^2\right\}.$$

# Example 1 – Power Analysis

Suppose you want to consider the power of superiority test based on the ratio in a balanced cross-over design with a Poisson count endpoint for sequence sample sizes between 500 and 1000. The superiority ratio is 1.2 and the actual ratio is 1.3, the fixed mean rate is estimated to be 1, and the period rate ratio is estimated to be between 0.9 and 1.1. The significance level is 0.05.

## Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab	
Solve For .....	<b>Power</b>
Higher Event Rates Are .....	<b>Better</b>
Alpha.....	<b>0.05</b>
n (Sample Size per Sequence) .....	<b>500 to 1000 by 100</b>
R0 (Superiority Ratio) .....	<b>1.2</b>
R1 (Actual Ratio) .....	<b>1.3</b>
Fixed Mean Rate ( $\mu$ ) .....	<b>1</b>
Period Rate Ratio (Rp).....	<b>0.9 1.0 1.1</b>

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## Output

Click the Calculate button to perform the calculations and generate the following output.

## Numeric Reports

## Numeric Results

Solve For: **Power**  
 Higher Proportions Are: Better  
 Hypotheses:  $H_0: R \leq R_0$  vs.  $H_1: R > R_0$

Power	Sample Size		Event Rate Ratio		Fixed Mean Rate $\mu$	Period Rate Ratio $R_p$	Alpha
	Sequence n	Total N	Superiority $R_0$	Actual $R_1$			
0.58213	500	1000	1.2	1.3	1	0.9	0.05
0.60184	500	1000	1.2	1.3	1	1.0	0.05
0.61901	500	1000	1.2	1.3	1	1.1	0.05
0.64956	600	1200	1.2	1.3	1	0.9	0.05
0.66994	600	1200	1.2	1.3	1	1.0	0.05
0.68750	600	1200	1.2	1.3	1	1.1	0.05
0.70771	700	1400	1.2	1.3	1	0.9	0.05
0.72799	700	1400	1.2	1.3	1	1.0	0.05
0.74529	700	1400	1.2	1.3	1	1.1	0.05
0.75742	800	1600	1.2	1.3	1	0.9	0.05
0.77704	800	1600	1.2	1.3	1	1.0	0.05
0.79357	800	1600	1.2	1.3	1	1.1	0.05
0.79958	900	1800	1.2	1.3	1	0.9	0.05
0.81812	900	1800	1.2	1.3	1	1.0	0.05
0.83356	900	1800	1.2	1.3	1	1.1	0.05
0.83511	1000	2000	1.2	1.3	1	0.9	0.05
0.85230	1000	2000	1.2	1.3	1	1.0	0.05
0.86643	1000	2000	1.2	1.3	1	1.1	0.05

Power The probability of rejecting a false null hypothesis when the alternative hypothesis is true.  
 n The sample size in each sequence (or group).  
 N The total sample size from both sequences. The sample is divided equally among sequences.  
 $R_0$  The superiority ratio used to specify the hypothesis test.  
 $R_1$  The actual event rate ratio at which power is calculated.  $R_1 = \lambda_t / \lambda_c$ .  
 $\mu$  The fixed mean rate of underlying random effects for the two treatments.  
 $R_p$  The rate ratio for Period 2 vs. Period 1 on a given subject, given a fixed treatment.  
 Alpha The probability of rejecting a true null hypothesis.

## Summary Statements

A 2x2 cross-over design (where higher hazard rates are considered to be better) will be used to test whether the treatment hazard rate ( $\lambda_t$ ) is superior to the control hazard rate ( $\lambda_c$ ) by a margin, with a superiority ratio of 1.2 ( $H_0: \lambda_t / \lambda_c \leq 1.2$  versus  $H_1: \lambda_t / \lambda_c > 1.2$ ). The comparison will be made using a one-sided Z-test based on the log of the event rate ratio, with a Type I error rate ( $\alpha$ ) of 0.05. The fixed mean rate of underlying random effects for the two treatments is assumed to be 1, and the rate ratio for Period 2 versus Period 1 on a given subject, given a fixed treatment, is assumed to be 0.9. To detect a hazard rate ratio ( $\lambda_t / \lambda_c$ ) of 1.3 with a sample size of 500 in each sequence (totaling 1000 subjects), the power is 0.58213.

## Superiority by a Margin Tests for the Ratio of Two Poisson Rates in a 2x2 Cross-Over Design

## Dropout-Inflated Sample Size

Dropout Rate	Sample Size		Dropout-Inflated Enrollment Sample Size		Expected Number of Dropouts	
	n	N	n'	N'	d	D
20%	500	1000	625	1250	125	250
20%	600	1200	750	1500	150	300
20%	700	1400	875	1750	175	350
20%	800	1600	1000	2000	200	400
20%	900	1800	1125	2250	225	450
20%	1000	2000	1250	2500	250	500

Dropout Rate	The percentage of subjects (or items) that are expected to be lost at random during the course of the study and for whom no response data will be collected (i.e., will be treated as "missing"). Abbreviated as DR.
n and N	The evaluable group and total sample sizes, respectively, at which power is computed (as entered by the user). If n subjects from each group are evaluated out of the n' subjects that are enrolled in the study, the design will achieve the stated power. $N = 2n$ .
n' and N'	The number of subjects that should be enrolled in the study in order to obtain n and N evaluable subjects, based on the assumed dropout rate. n' is calculated by inflating n using the formula $n' = n / (1 - DR)$ , with n' always rounded up. (See Julious, S.A. (2010) pages 52-53, or Chow, S.C., Shao, J., Wang, H., and Lokhnygina, Y. (2018) pages 32-33.). $N' = 2n'$ .
d and D	The expected number of group and total dropouts, respectively. $d = n' - n$ and $D = 2d$ .

## Dropout Summary Statements

Anticipating a 20% dropout rate, 625 subjects should be enrolled in each group to obtain final sample sizes of 500 subjects per group.

## References

- Lui, Kung-Jong. 2016. Crossover Designs: Testing, Estimation, and Sample Size. John Wiley & Sons Ltd. Chichester, West Sussex, England.
- Lui, Kung-Jong. 2013. Sample size determination for testing equality in Poisson frequency data under an AB/BA crossover trial. Pharmaceutical Statistics. Volume 12, pages 74-81.

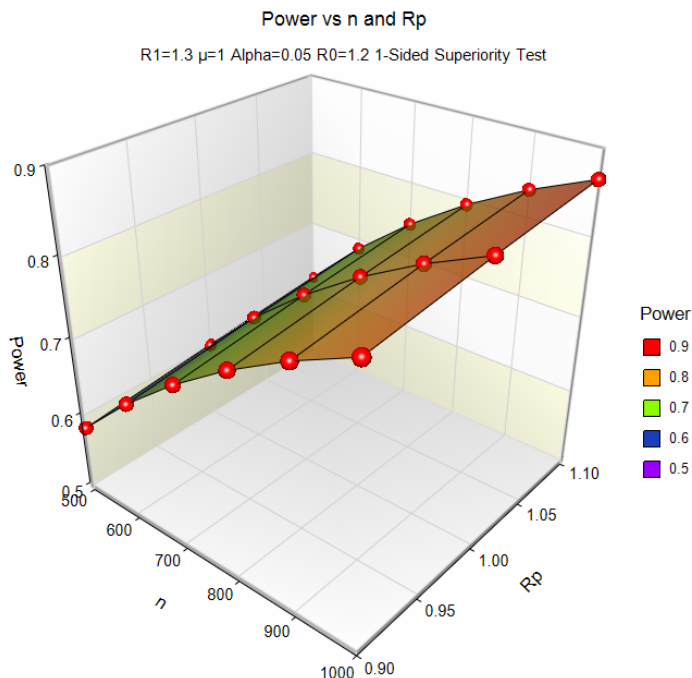
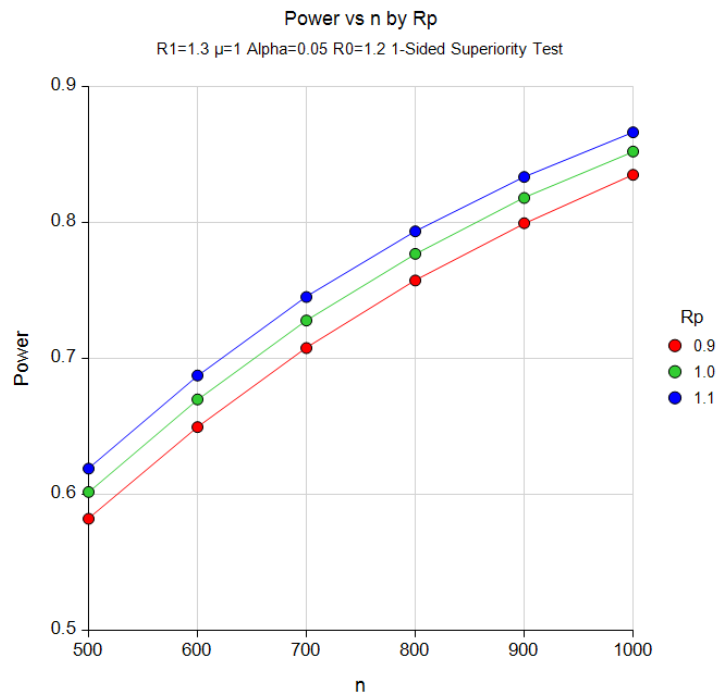
This report shows the values of each of the parameters, one scenario per row.



## Superiority by a Margin Tests for the Ratio of Two Poisson Rates in a 2x2 Cross-Over Design

## Plots Section

## Plots



These plots show the relationship between sample size, Rp, and power. We see that sample sizes of between 800 and 900 per sequence are required to detect an actual event rate ratio of 1.3 with 90% power when the superiority bound ratio is 1.2.

## Example 2 – Calculating Sample Size (Validation using Hand Calculations)

In this example, we'll demonstrate how to compute sample size for a superiority by a margin test of two Poisson rates from a 2x2 cross-over design. This example will also serve as validation for this procedure. We couldn't find any published examples of this test, so we'll validate the procedure by hand. Let's find the sample size required to detect and actual event rate ratio of 1.3 with 80% power at a significance level of 0.05 when the superiority ratio is 1.2 and both the fixed mean rate and period rate ratio are equal to 1. These values are similar to those used in Table II on page 78 of Lui (2013) for a test of inequality.

First, we need to compute the variance component with

$$\begin{aligned}
 p_1 &= \frac{e^{\eta+\gamma}}{1 + e^{\eta+\gamma}} = \frac{e^{\eta}e^{\gamma}}{1 + e^{\eta}e^{\gamma}} = \frac{1.3}{1 + 1.3} = 0.5652 \\
 p_2 &= \frac{e^{\gamma}}{e^{\eta} + e^{\gamma}} = \frac{1}{1.3 + 1} = 0.4348 \\
 V(\mu, \eta, \gamma) &= \frac{1}{4} \left( \frac{1}{\mu(1 + e^{\eta+\gamma})p_1(1 - p_1)} + \frac{1}{\mu(e^{\eta} + e^{\gamma})p_2(1 - p_2)} \right) \\
 &= \frac{1}{4} \left( \frac{1}{1(1 + 1.3)0.5652(1 - 0.5652)} + \frac{1}{1(1.3 + 1)0.4348(1 - 0.4348)} \right) \\
 &= \frac{1}{4} \left( \frac{1}{0.5652} + \frac{1}{0.5652} \right) = 0.8846
 \end{aligned}$$

The formula for sample size given in Lui (2016) on page 87 is

$$\begin{aligned}
 n &= \text{Ceiling} \left\{ \left( \frac{(Z_{1-\alpha} + Z_{1-\beta})\sqrt{V(\mu, \eta, \gamma)}}{\log(R_1) - \log(R_0)} \right)^2 \right\} \\
 &= \text{Ceiling} \left\{ \left( \frac{(1.6449 + 0.8416)\sqrt{0.8846}}{\log(1.3) - \log(1.2)} \right)^2 \right\} \\
 &= \text{Ceiling}\{853.6485\} = 854.
 \end{aligned}$$

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## Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

### Design Tab

Solve For ..... **Sample Size**  
 Higher Event Rates Are ..... **Better**  
 Power..... **0.80**  
 Alpha..... **0.05**  
 R0 (Superiority Ratio) ..... **1.2**  
 R1 (Actual Ratio) ..... **1.3**  
 Fixed Mean Rate ( $\mu$ ) ..... **1**  
 Period Rate Ratio (Rp)..... **1**

## Output

Click the Calculate button to perform the calculations and generate the following output.

### Numeric Results

Solve For: **Sample Size**  
 Higher Proportions Are: **Better**  
 Hypotheses: **H0:  $R \leq R_0$  vs. H1:  $R > R_0$**

Power	Sample Size		Event Rate Ratio		Fixed Mean Rate $\mu$	Period Rate Ratio Rp	Alpha
	Sequence n	Total N	Superiority R0	Actual R1			
0.80014	854	1708	1.2	1.3	1	1	0.05

The sample size of 854 per sequence calculated by **PASS** matches our hand calculations.