Chapter 405

Tests for One Exponential Mean

Introduction

This program module designs studies for testing hypotheses about the mean of the exponential distribution. Such tests are often used in *reliability acceptance testing*, also called *reliability demonstration testing*.

Results are calculated for plans that are *time censored* or *failure censored*, as well as for plans that use *with replacement* or *without replacement* sampling. We adopt the basic methodology outlined in Epstein (1960), Juran (1979), Bain and Engelhardt (1991), and Schilling (1982).

Technical Details

The test procedures described here assume that lifetimes follow the exponential distribution. The density of the exponential distribution is written as

$$f(t) = \frac{1}{\theta}\exp\left(-\frac{t}{\theta}\right)$$

The parameter θ is interpreted as average failure time, mean time to failure (MTTF), or mean time between failures (MTBF). Its reciprocal is the failure rate.

The reliability, or probability that a unit continues running beyond time *t*, is

$$R(t) = e^{-\frac{t}{\theta}}$$

Hypothesis Test

The relevant statistical hypothesis is $H_0: \theta_0 = \theta_1$ versus the one-sided alternative $H_1: \theta_0 > \theta_1$. Here, θ_0 represents an acceptable (high) mean life usually set from the point of view of the producer and θ_1 represents some unacceptable (low) mean life usually set from the point of view of the consumer. The test procedure is to reject the null hypothesis if the observed mean life $\hat{\theta}$ is larger than a critical value selected to meet the error rate criterion.

The error rates are often interpreted in reliability testing as *risks*. The *consumer* runs the risk that the study will fail to reject products that have a reliability less than they have specified. This *consumer risk* is β. Similarly, the *producer* runs the risk that the study will reject products that actually meet the consumer's requirements. This *producer risk* is α.

Fixed-Failure Sampling Plans

Fixed failure plans are those in which a specified number of items, *n*, are observed until a specified number of items, r_0, fail. The length of the study t_0 is random. Failed items may, or may not, be immediately replaced (*with replacement* versus *without replacement*).

The test statistic is the observed mean life $\hat{\theta}$ which is computed using

$$\hat{\theta} = \frac{\sum_{i=\text{all test items}} t_i}{r_0}$$

where t_i is the elapsed time that the *i*th item is tested, whether measured until failure or until the study is completed.

For both with-replacement and without-replacement sampling, $\hat{\theta}$ follows the two-parameter gamma distribution with density

$$g(y|r_0, \theta) = \frac{1}{(r_0 - 1)!}\left(\frac{r_0}{\theta}\right)^r y^{r_0 - 1} e^{-r_0 y/\theta}$$

This may be converted to a standard, one-parameter gamma using the transformation

$$x = r_0 y/\theta$$

However, because chi-square tables were more accessible, and because the gamma distribution may be transformed to the chi-square distribution, most results in the statistical literature are based on the chi-square distribution. That is, $2r_0\hat{\theta}/\theta$ is distributed as a chi-square random variable with $2r_0$ degrees of freedom.

Assuming that the testing of all *n* items begins at the same instant, the expected length of time needed to observe the first r_0 failures is

$$E(t_0) = \begin{cases} \theta \displaystyle\sum_{i=1}^{r_0} \frac{1}{n - i + 1} & \text{without replacement} \\ \dfrac{\theta r_0}{n} & \text{with replacement} \end{cases}$$

If you choose to solve the without replacement equation for *n*, you can make use of the approximation

$$\sum_{i=1}^{r} \frac{1}{n - i + 1} \approx \ln\left(\frac{n + 0.5}{n - r + 0.5}\right)$$

Using the above results, sampling plans that meet the specified producer and consumer risk values may be found using the result (see Epstein (1960) page 437) that r_0 is the smallest integer such that

$$\frac{\chi^2_{\alpha,2r_0}}{\chi^2_{1-\beta,2r_0}} \geq \frac{\theta_1}{\theta_0} \text{ for testing } H_1: \theta_0 > \theta_1$$

and

$$\frac{\chi^2_{\beta,2r_0}}{\chi^2_{1-\alpha,2r_0}} \geq \frac{\theta_0}{\theta_1} \text{ for testing } H_1: \theta_0 < \theta_1$$

Note that the above formulation depends on r_0 but not *n*. An appropriate value of *n* can be found by considering $E(t_0)$. Two options are available.

1. The value of *n* is set (perhaps on economic grounds) and the value of $E(t_0)$ is calculated.
2. The value of $E(t_0)$ is set and the value of *n* is calculated.

Fixed-Time Sampling Plans

Fixed Time plans refer to those in which a specified number of items *n* are observed for a fixed length of time t_0. The number of items failing *r* is recorded. Sampling can be with or without replacement. The accept/reject decision can be based on *r* or the observed mean life $\hat{\theta}$ which is computed using

$$\hat{\theta} = \frac{\sum_{i=\text{all test items}} t_i}{r}$$

where t_i is the time that the *i*th item is being tested, whether measured until failure or until the study is completed.

With Replacement Sampling

If failed items are immediately replaced with additional items, the distribution of *r* (and $\hat{\theta}$, since $\hat{\theta} = nt_0 / r$) follows the Poisson distribution. The probability distribution of *r* is given by the Poisson probability formula

$$P(r \leq r_0 | r, \theta) = \sum_{i=0}^{r} \frac{(nt_0 / \theta)^i}{i!} e^{-nt_0/\theta}$$

Thus, values of *n* and t_0 can be found which meet the α and β requirements.

Without Replacement Sampling

If failed items are not replaced, the distributions of *r* and $\hat{\theta}$ are different and thus the power and sample size calculations depend on which statistic will be used. The probability distribution of *r* is given by the binomial formula

$$P(r \le r_0 | r, \theta) = \sum_{i=0}^{r} \binom{n}{i} p^i (1-p)^{n-i}$$

where

$$p = 1 - e^{-t_0/\theta}$$

Thus, values of *n* and t_0 can be found which meet the α and β requirements. Note that this formulation ignores the actual failure times.

If $\hat{\theta}$ will be used as the test statistic, power calculations must be based on it. Bartholomew (1963) gave the following results for the case *r* > 0.

$$\Pr(\hat{\theta} \ge \theta_C) = \frac{1}{1 - e^{-nt_0/\theta}} \sum_{k=1}^{n} \binom{n}{k} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \exp\left\{-\frac{t_0}{\theta}(n-k+i)\right\} \int_W^\infty g(x)\mathrm{dx}$$

where *g*(*x*) is the chi-square density function with 2*k* degrees of freedom and

$$W = \frac{2k}{\theta}\left\langle \theta_C - \frac{t_0}{k}(n-k+i)\right\rangle$$

$$\langle X \rangle = \begin{cases} X & \text{if } X > 0 \\ 0 & \text{otherwise} \end{cases}$$

The above equation is numerically unstable for large values of *N*, so we use the following approximation also given by Bartholomew (1963). This approximation is used when *N* > 30 or when the exact equation cannot be calculated. Bain and Engelhardt (1991) page 140 suggest that this normal approximation can be used when *p* > 0.5

$$z = \frac{u\sqrt{np}}{\sqrt{1 - \frac{2u(1-p)\log_e(1-p)}{p} + (1-p)u^2}}$$

where

$$u = \frac{\hat{\theta} - \theta}{\theta}$$

$$p = 1 - e^{-t_0/\theta}$$

Example 1 – Power for Several Sample Sizes

This example will calculate power for a time terminated, without replacement study in which the results will be analyzed using theta-hat. The study will be used to test the alternative hypothesis that Theta0 > Theta1, where Theta0 = 2.0 days and Theta1 = 1.0 days. The test duration is 1.0 days. Funding for the study will allow for a sample size of up to 40 test items. The researchers decide to look at sample sizes of 10, 20, 30, and 40. Significance levels of 0.01 and 0.05 will be considered.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For	**Power**
Alternative Hypothesis	**Ha: θ1 < θ0**
Replacement Method	**Without Replacement**
Termination Criterion	**Fixed Time using Θhat**
t0 (Test Duration Time)	**1**
Alpha (Producer's Risk)	**0.01 0.05**
N (Sample Size)	**10 to 40 by 10**
θ0 (Baseline Mean Life)	**2**
θ1 (Alternative Mean Life)	**1**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Reports

Numeric Results

Solve For: Power
Sampling: Without Replacement
Test Based On: ϴhat with Fixed Running Time, t0
Hypotheses: H0: ϴ = ϴ0 vs. Ha: ϴ < ϴ0
(Reject H0 if ϴhat ≤ ϴC)

			Mean Life			Alpha	
Power	Sample Size N	Time t0	Baseline ϴ0	Alternative ϴ1	Boundary ϴC	Target	Actual
0.21695	10	1	2	1	0.7	0.01	0.01
0.45485	20	1	2	1	1.0	0.01	0.01
0.67159	30	1	2	1	1.1	0.01	0.01
0.80628	40	1	2	1	1.2	0.01	0.01
0.46940	10	1	2	1	1.0	0.05	0.05
0.71828	20	1	2	1	1.2	0.05	0.05
0.86665	30	1	2	1	1.3	0.05	0.05
0.93730	40	1	2	1	1.4	0.05	0.05

Power The probability of rejecting a false null hypothesis when the alternative hypothesis is true.
N The size of the sample drawn from the population.
t0 The test duration time. It provides the scale for ϴ0 and ϴ1.
ϴ0 The Mean Life under the null hypothesis.
ϴ1 The Mean Life under the alternative hypothesis.
ϴC The critical value boundary of mean life.
Alpha The probability of rejecting a true null hypothesis.

Summary Statements

A single-group design will be used to test whether the mean lifetime (θ) is less than 2 (H0: θ ≥ 2 versus H1: θ < 2, reject H0 if ϴhat ≤ ϴC). The comparison will be made using a one-sided, one-sample Exponential mean test based on a fixed elapsed time and using the ϴhat (observed mean life) test statistic, with a Type I error rate (α) of 0.01. In this design, failed items are not replaced with new items. The study is terminated when it has run for 1 time unit. To detect a mean lifetime of 1 with a sample size of 10 units, the power is 0.21695.

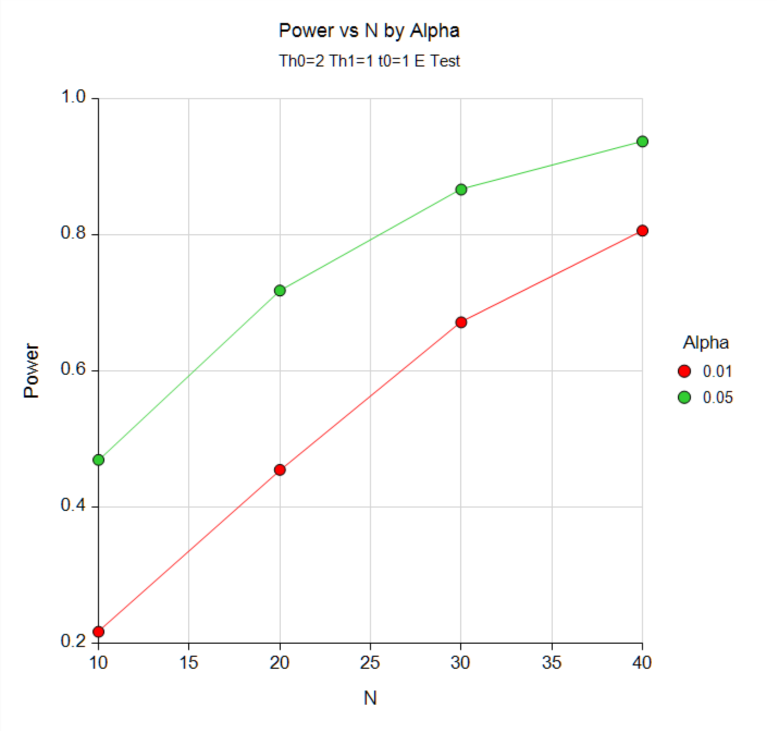
References

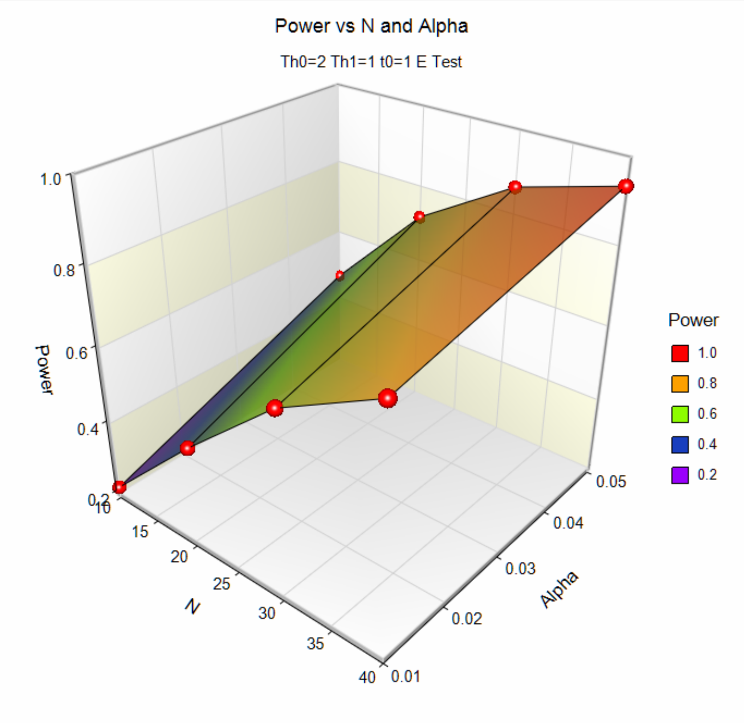
Bain, L.J. and Engelhardt, M. 1991. Statistical Analysis of Reliability and Life-Testing Models. Marcel Dekker. New York.
Epstein, Benjamin. 1960. 'Statistical Life Test Acceptance Procedures.' Technometrics. Volume 2.4, pages 435-446.
Juran, J.M. 1979. Quality Control Handbook. McGraw-Hill. New York.
Schilling, Edward. 1982. Acceptance Sampling in Quality Control. Marcel-Dekker. New York.

This report shows the power for each of the scenarios. The critical value, Theta C, is also provided.

Plots Section

Plots





These plots show the relationship between power and sample size.

Example 2 – Validation using Epstein (1960)

Epstein (1960), page 438, presents a table giving values of *r* necessary to meet risk criteria for various values of alpha, beta, theta0, and theta1 for the fixed failures case. Specifically, when θ0 = 5, θ1 = 2, beta = 0.05, and alpha = 0.01, 0.05, and 0.10, he finds *r* = 21, 14, and 11. We will now duplicate these results.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For ....**Sample Size**
Alternative Hypothesis ....**Ha: θ0 > θ1**
Replacement Method......................................**Without Replacement**
Termination Criterion ....**Fixed Failures, Fixed E(t0)**
t0 (Test Duration Time)...................................**1**
E(t0) based on θ1 ....**Unchecked**
Power = 1-Beta (Beta is Consumer's Risk)......**0.95**
Alpha (Producer's Risk) **0.01 0.05 0.1**
θ0 (Baseline Mean Life)..................................**5**
θ1 (Alternative Mean Life)...............................**2**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Solve For: Sample Size
Sampling: Without Replacement
Test Based On: Fixed Failures, r, and Fixed Expected Time, E(t0)
Hypotheses: H0: Θ = Θ0 vs. Ha: Θ < Θ0
(Reject H0 if r ≥ r0)

				Mean Life		Alpha	
Power	Number of Failures r0	Sample Size N	Time E(t0)	Baseline Θ0	Alternative Θ1	Target	Actual
0.95841	21	115	1	5	2	0.01	0.01
0.95956	14	77	1	5	2	0.05	0.05
0.96221	11	60	1	5	2	0.10	0.10

PASS calculates 21, 14, and 11 for *r* as in Epstein.

We should note that occasionally our results differ from those of Epstein. We have checked a few of these carefully by hand, and in every case, we have found our results to be correct.