

## Chapter 412

# Tests for One Poisson Rate

## Introduction

The Poisson probability law gives the probability distribution of the number of events occurring in a specified interval of time or space. The Poisson distribution is often used to fit count data, such as the number of defects on an item, the number of accidents at an intersection during a year, the number of calls to a call center during an hour, or the number of meteors seen in the evening sky during an hour.

The Poisson distribution is characterized by a single parameter,  $\lambda$ , which is the mean number of occurrences during the interval. This procedure calculates the power or sample size for testing whether  $\lambda$  is less than or greater than a specified value. This test is usually called the *test of the Poisson rate (or mean)*.

The test is described in Ostle and Malone (1988) and the power calculation is given in Guenther (1977).

## Test Procedure

Assume that the mean rate is  $\lambda_0$ . To test  $H_0 : \lambda \leq \lambda_0$  vs.  $H_a : \lambda > \lambda_0$ , you would take the following steps.

1. **Find the critical value.** Choose the critical value  $X^*$  so that the probability of rejecting  $H_0$  when it is true is equal to  $\alpha$ . This is done by solving the following inequality for  $X^*$ .

$$\sum_{x=X^*}^{\infty} e^{-n\lambda_0} \frac{(n\lambda_0)^x}{x!} \leq \alpha.$$

Note that because  $X$  is an integer, equality will seldom occur. Therefore, the minimum value of  $X^*$  is found for which the inequality holds.

2. **Select a sample of  $n$  items compute the total number of events  $X = \sum_{i=1}^n x_i$ .** If  $X > X^*$  reject  $H_0$  in favor of  $H_a$ .

The test in the other direction ( $H_0 : \lambda \geq \lambda_0$  vs.  $H_a : \lambda < \lambda_0$ ) is computed similarly.

## Tests for One Poisson Rate

### Assumptions

The assumptions of the one-sample *Poisson* test are:

1. The data are counts (discrete) that follow the Poisson distribution.
2. The sample is a simple random sample from its population. Each individual in the population has an equal probability of being selected in the sample.

### Limitations

There are few limitations when using these tests. As long as the assumption that the mean occurrence rate is constant is met, the test is valid.

### Technical Details

#### Computing Power

The power is computing for a specific alternative value  $\lambda_1$  using the following formula.

$$\begin{aligned} \text{Power} &= 1 - \beta \\ &= \sum_{x=X^*}^{\infty} e^{-n\lambda_1} \frac{(n\lambda_1)^x}{x!} \end{aligned}$$

#### Computing Sample Size

Following Guenther (1977), the sample size,  $n$ , is found by increasing the value of  $d$  in the following expression until the left-hand end-point is less than the right-hand end-point and the interval contains at least one integer.

$$\frac{X_{2d;1-\beta}^2}{2\lambda_1} \leq n \leq \frac{X_{2d;\alpha}^2}{2\lambda_0}, \quad d = 1, 2, 3, \dots$$

Here  $X_{v;p}^2$  is a percentage point of the chi-square distribution with  $v$  degrees of freedom.

### Procedure Options

This section describes the options that are specific to this procedure. These are located on the Design tab. For more information about the options of other tabs, go to the Procedure Window chapter.

#### Design Tab

The Design tab contains most of the parameters and options that you will be concerned with.

#### Solve For

##### Solve For

This option specifies the parameter to be calculated from the values of the other parameters. Under most conditions, you would select either *Power* or *Sample Size*.

## Tests for One Poisson Rate

Select *Sample Size* when you want to determine the sample size needed to achieve a given power and alpha error level.

Select *Power* when you want to calculate the power of an experiment that has already been run.

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### Test

#### Alternative Hypothesis

Specify the one-sided, alternative hypothesis of the test.

The choices are:

1.  $H_a : \lambda_1 > \lambda_0$ . Upper, one-sided test.
2.  $H_a : \lambda_1 < \lambda_0$ . Lower, one-sided test.

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### Power and Alpha

#### Power

This option specifies one or more values for power. Power is the probability of rejecting a false null hypothesis, and is equal to one minus beta. Beta is the probability of a type-II error, which occurs when a false null hypothesis is not rejected.

Values must be between zero and one. Historically, the value of 0.80 (beta = 0.20) was used for power. Now, 0.90 (beta = 0.10) is also commonly used.

A single value may be entered here or a range of values such as *0.8 to 0.95 by 0.05* may be entered.

#### Alpha

This option specifies one or more values for the probability of a type-I error. A type-I error occurs when a true null hypothesis is rejected.

Values must be between zero and one. For one-sided tests such as this, the value of 0.025 is recommended for alpha. You may enter a range of values such as *0.025 0.05 0.10* or *0.025 to 0.05 by 0.005*.

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### Sample Size

#### n (Sample Size)

This option specifies one or more values of the sample size, the number of individuals in the study. This value must be an integer greater than one. Note that you may enter a list of values using the syntax *50,100,150,200,250* or *50 to 250 by 50*.

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### Effect Size – Means

#### $\lambda_0$ (Null or Baseline Rate)

This option specifies one or more values of the mean occurrence rate corresponding to the null hypothesis. This value must be greater than zero and not equal to  $\lambda_1$ .

#### $\lambda_1$ (Alternative Rate)

This option specifies one or more values of the mean occurrence rate corresponding to the alternative hypothesis. This value must be greater than zero and not equal to  $\lambda_0$ .

## Tests for One Poisson Rate

## Example 1 – Power after a Study

This example demonstrates how to calculate the power for specific values of the other parameters. Suppose that accidents have occurred at an intersection at an average rate of 1 per month for the last several years. Recently, a distraction has been constructed near the intersection that appears to have increased the accident rate. Suppose the sample sizes are 12 and 24 months and alpha is 0.025. What is the power to detect alternatives of 1.1, 1.5, 2.0, and 2.5?

### Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Tests for One Poisson Rate** procedure window. You may then make the appropriate entries as listed below, or open **Example 1** by going to the **File** menu and choosing **Open Example Template**.

<u>Option</u>	<u>Value</u>
<b>Design Tab</b>	
Solve For .....	<b>Power</b>
Alternative Hypothesis .....	<b>Ha: <math>\lambda_1 &gt; \lambda_0</math></b>
Alpha.....	<b>0.025</b>
n (Sample Size) .....	<b>12 24</b>
$\lambda_0$ (Null or Baseline Rate).....	<b>1.0</b>
$\lambda_1$ (Alternative Rate).....	<b>1.1 1.4 1.8 2.2 2.5</b>

### Annotated Output

Click the Calculate button to perform the calculations and generate the following output.

### Numeric Results

#### Numeric Results for a One-Sample Poisson Rate Test

Null Hypothesis:  $\lambda_1 \leq \lambda_0$  Alternative Hypothesis:  $\lambda_1 > \lambda_0$

Power	n	Target Alpha	Actual Alpha	$\lambda_0$	$\lambda_1$	Diff ( $\lambda_1 - \lambda_0$ )	Effect Size	Beta
0.0484	12	0.0250	0.0213	1.00	1.10	0.10	0.0953	0.9516
0.0623	24	0.0250	0.0206	1.00	1.10	0.10	0.0953	0.9377
0.2476	12	0.0250	0.0213	1.00	1.40	0.40	0.3381	0.7524
0.4273	24	0.0250	0.0206	1.00	1.40	0.40	0.3381	0.5727
0.6638	12	0.0250	0.0213	1.00	1.80	0.80	0.5963	0.3362
0.9108	24	0.0250	0.0206	1.00	1.80	0.80	0.5963	0.0892
0.9154	12	0.0250	0.0213	1.00	2.20	1.20	0.8090	0.0846
0.9962	24	0.0250	0.0206	1.00	2.20	1.20	0.8090	0.0038
0.9781	12	0.0250	0.0213	1.00	2.50	1.50	0.9487	0.0219
0.9998	24	0.0250	0.0206	1.00	2.50	1.50	0.9487	0.0002

#### References

- Guenther, William C. 1977. Sampling Inspection in Statistical Quality Control. Griffin's Statistical Monographs. Macmillan, NY. Pages 25-29.
- Ostle, B. and Malone, L. 1988. Statistics in Research, 4th Edition. Iowa State University Press. Iowa. Pages 116-118.

### Tests for One Poisson Rate

**Report Definitions**

Power is the probability of rejecting a false null hypothesis. It should be close to one.  
 n is the size of the sample drawn from the population. To conserve resources, it should be small.  
 Alpha is the probability of rejecting a true null hypothesis. It should be small.  
 $\lambda_0$  is the value of the population mean rate under the null hypothesis.  
 $\lambda_1$  is the value of the population mean rate under the alternative hypothesis.  
 Diff is the value of  $\lambda_1 - \lambda_0$ , the difference being tested.  
 Effect Size is the value of  $(\lambda_1 - \lambda_0) / \sqrt{\lambda_1}$ .  
 Beta is the probability of accepting a false null hypothesis. It should be small.

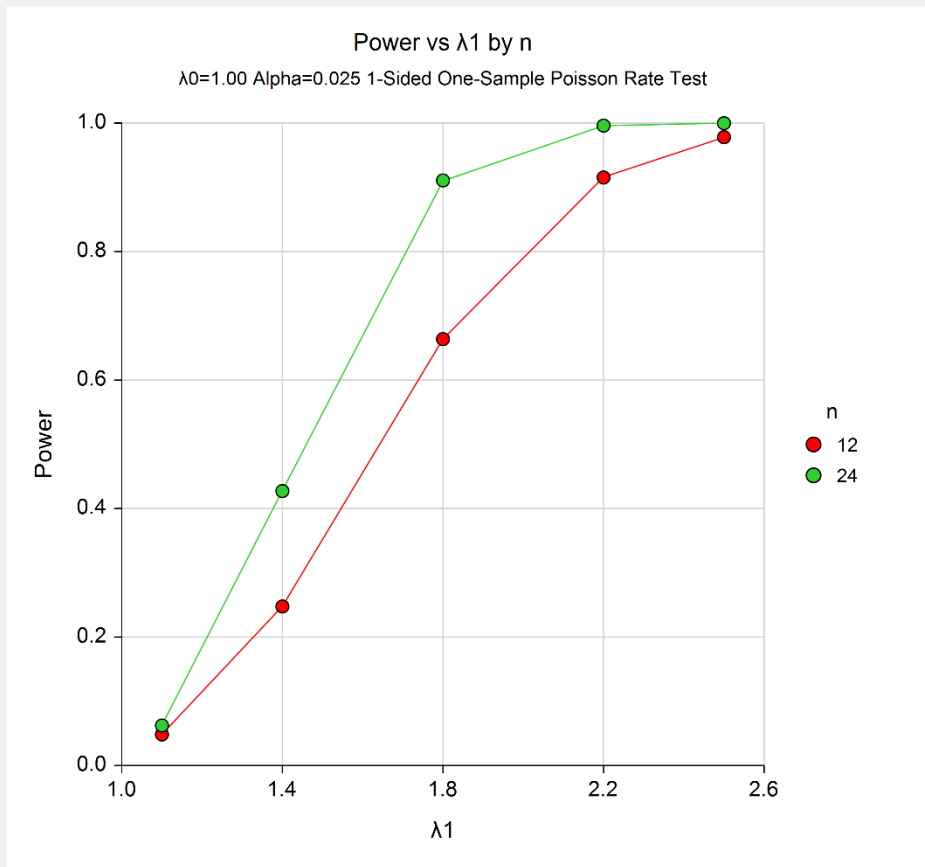
**Summary Statements**

A sample size of 12 achieves 5% power to detect a difference of 0.10 between the alternative hypothesis mean rate of 1.10 and the null hypothesis mean rate of 1.00 with a significance level (alpha) of 0.0250 using a one-sided, one-sample Poisson rate test.

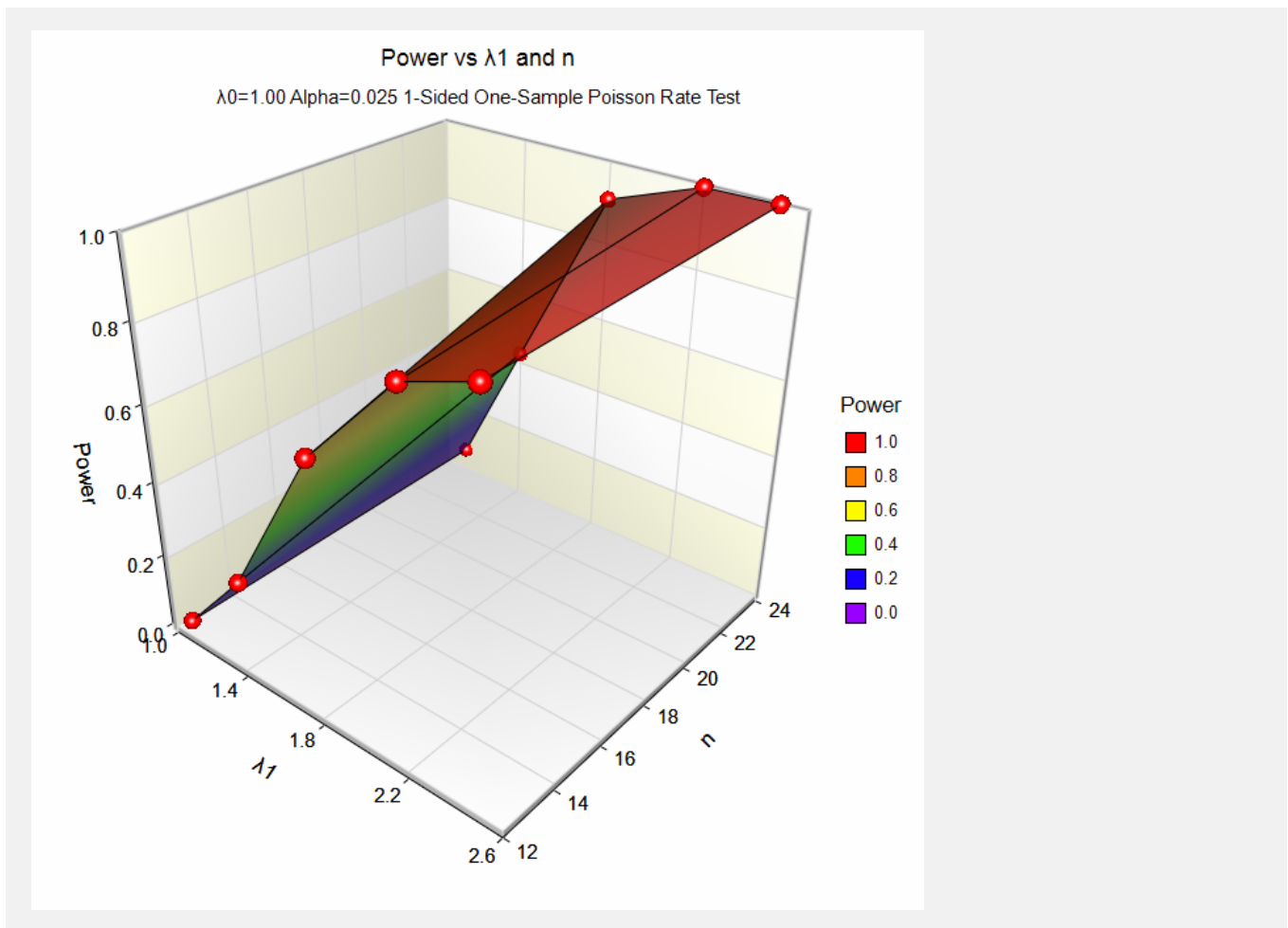
This report shows the values of each of the parameters, one scenario per row. The values of power and beta were calculated from the other parameters.

Note that the actual power achieved is greater than the target power. Similarly, the actual alpha is less than the target alpha. These differences occur because only integer values of the count variable occur.

### Plots Section



Tests for One Poisson Rate



These plots show the relationship between sample size and power for various values of the alternative rate and the sample size.

## Tests for One Poisson Rate

## Example 2 – Finding the Sample Size

This example will extend Example 1 to the case in which we want to find the necessary sample size to achieve at least 90% power. This is done as follows.

### Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Tests for One Poisson Rate** procedure window. You may then make the appropriate entries as listed below, or open **Example 2** by going to the **File** menu and choosing **Open Example Template**.

<u>Option</u>	<u>Value</u>
<b>Design Tab</b>	
Solve For .....	<b>Sample Size</b>
Alternative Hypothesis .....	<b>Ha: <math>\lambda_1 &gt; \lambda_0</math></b>
Power .....	<b>0.90</b>
Alpha .....	<b>0.025</b>
$\lambda_0$ (Null or Baseline Rate) .....	<b>1.0</b>
$\lambda_1$ (Alternative Rate) .....	<b>1.1 1.4 1.8 2.2 2.5</b>

### Output

Click the Calculate button to perform the calculations and generate the following output.

### Numeric Results

Numeric Results for a One-Sample Poisson Rate Test								
Null Hypothesis: $\lambda_1 \leq \lambda_0$ Alternative Hypothesis: $\lambda_1 > \lambda_0$								
Power	n	Target Alpha	Actual Alpha	$\lambda_0$	$\lambda_1$	Diff ( $\lambda_1 - \lambda_0$ )	Effect Size	Beta
0.9002	1100	0.0250	0.0250	1.00	1.10	0.10	0.0953	0.0998
0.9009	80	0.0250	0.0220	1.00	1.40	0.40	0.3381	0.0991
0.9108	24	0.0250	0.0206	1.00	1.80	0.80	0.5963	0.0892
0.9154	12	0.0250	0.0213	1.00	2.20	1.20	0.8090	0.0846
0.9366	9	0.0250	0.0220	1.00	2.50	1.50	0.9487	0.0634

This report shows the sample sizes that are necessary to achieve the required power.

## Tests for One Poisson Rate

## Example 3 – Finding the Minimum Detectable Difference

Continuing with the previous example, suppose only 10 months of data are available. What is the minimum detectable difference that can be detected by this design?

### Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Tests for One Poisson Rate** procedure window. You may then make the appropriate entries as listed below, or open **Example 3** by going to the **File** menu and choosing **Open Example Template**.

<u>Option</u>	<u>Value</u>
<b>Design Tab</b>	
Solve For .....	$\lambda_1$
Alternative Hypothesis .....	$H_a: \lambda_1 > \lambda_0$
Power .....	<b>0.90</b>
Alpha .....	<b>0.025</b>
n (Sample Size) .....	<b>10</b>
$\lambda_0$ (Null or Baseline Rate) .....	<b>1.0</b>

### Output

Click the Calculate button to perform the calculations and generate the following output.

### Numeric Results

Numeric Results for a One-Sample Poisson Rate Test								
Null Hypothesis: $\lambda_1 \leq \lambda_0$ Alternative Hypothesis: $\lambda_1 > \lambda_0$								
Power	n	Target Alpha	Actual Alpha	$\lambda_0$	$\lambda_1$	Diff ( $\lambda_1 - \lambda_0$ )	Effect Size	Beta
0.9000	10	0.0250	0.0143	1.00	2.36	1.36	0.8856	0.1000

This report shows that the minimum detectable difference is  $2.36 - 1.00 = 1.36$ .



## Tests for One Poisson Rate

**Example 4 – Validation using Guenther (1977)**

Guenther (1977) page 27 gives an example in which  $\lambda_0 = 0.05$ ,  $\lambda_1 = .2$ ,  $\alpha = 0.05$ ,  $\beta = 0.10$ , and  $n = 47$ . We will now run this example.

**Setup**

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Tests for One Poisson Rate** procedure window. You may then make the appropriate entries as listed below, or open **Example 4** by going to the **File** menu and choosing **Open Example Template**.

<u>Option</u>	<u>Value</u>
<b>Design Tab</b>	
Solve For .....	<b>Sample Size</b>
Alternative Hypothesis .....	<b>Ha: <math>\lambda_1 &gt; \lambda_0</math></b>
Power .....	<b>0.90</b>
Alpha .....	<b>0.025</b>
$\lambda_0$ (Null or Baseline Rate) .....	<b>0.05</b>
$\lambda_1$ (Alternative Rate) .....	<b>0.20</b>

**Output**

Click the Calculate button to perform the calculations and generate the following output.

**Numeric Results**

Numeric Results for a One-Sample Poisson Rate Test								
Null Hypothesis: $\lambda_1 \leq \lambda_0$ Alternative Hypothesis: $\lambda_1 > \lambda_0$								
Power	n	Target Alpha	Actual Alpha	$\lambda_0$	$\lambda_1$	Diff ( $\lambda_1 - \lambda_0$ )	Effect Size	Beta
0.9065	47	0.0500	0.0327	0.05	0.20	0.15	0.3354	0.0935

Note that the value of  $n$  is indeed 47.