Chapter 650

Tests for One Variance

Introduction

Occasionally, researchers are interested in the estimation of the variance (or standard deviation) rather than the mean. This module calculates the sample size and performs power analysis for hypothesis tests concerning a single variance.

Technical Details

If a variable *X* is normally distributed with mean μ and variance σ^2, the sample variance is distributed as a Chi-square random variable with *N* - 1 degrees of freedom, where *N* is the sample size. That is,

$$\chi^2 = \frac{(N-1)s^2}{\sigma^2}$$

is distributed as a Chi-square random variable. The sample statistic, s^2, is calculated as

$$s^2 = \frac{\sum_{i=1}^{N}(X_i - \bar{X})^2}{N-1}.$$

If σ_1^2 is the assumed actual value of the variance under the alternative hypothesis, then the power or sample size of a hypothesis test about the variance can be calculated using the appropriate one of the following three formulas from Ostle and Malone (1988) page 130.

Case 1: $H_0: \sigma^2 = \sigma_0^2$ versus $H_a: \sigma^2 \neq \sigma_0^2$

$$\beta = P\left(\frac{\sigma_0^2}{\sigma_1^2}\chi^2_{\alpha/2,N-1} < \chi^2 < \frac{\sigma_0^2}{\sigma_1^2}\chi^2_{1-\alpha/2,N-1}\right)$$

Case 2: $H_0: \sigma^2 \leq \sigma_0^2$ versus $H_a: \sigma^2 > \sigma_0^2$

$$\beta = P\left(\chi^2 < \frac{\sigma_0^2}{\sigma_1^2}\chi^2_{1-\alpha,N-1}\right)$$

Case 3: $H_0: \sigma^2 \geq \sigma_0^2$ versus $H_a: \sigma^2 < \sigma_0^2$

$$\beta = P\left(\chi^2 > \frac{\sigma_0^2}{\sigma_1^2}\chi^2_{\alpha,N-1}\right)$$

Example 1 – Calculating the Power

A machine used to perform a particular analysis is to be replaced with a new type of machine if the new machine reduces the variation in the output. The current machine has been tested repeatedly and found to have an output variance of 42.5. The new machine will be cost effective if it can reduce the variance by 30% to 29.75. If the significance level is set to 0.05, calculate the power for sample sizes of 10, 50, 90, 130, 170, 210, and 250.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For .... **Power**
Alternative Hypothesis .... **Ha: σ² < σ0²**
Known Mean... **Unchecked**
Alpha.. **0.05**
N (Sample Size).. **10 50 90 130 170 210 250**
Input Type... **Variances**
σ0² (Baseline Variance) **42.5**
σ1² (Alternative Variance)............................. **29.75**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Reports

Numeric Results

Solve For: Power
Hypotheses: H0: σ² ≥ σ0² vs. Ha: σ² < σ0²
Mean: Not Known (Chi-Square df = N - 1)

	Sample	Variance		
	Size	Baseline	Alternative	
Power	N	σ0²	σ1²	Alpha
0.14448	10	42.5	29.75	0.05
0.50556	50	42.5	29.75	0.05
0.74775	90	42.5	29.75	0.05
0.88174	130	42.5	29.75	0.05
0.94785	170	42.5	29.75	0.05
0.97806	210	42.5	29.75	0.05
0.99111	250	42.5	29.75	0.05

Power	The probability of rejecting a false null hypothesis when the alternative hypothesis is true.
N	The size of the sample drawn from the population.
$\sigma0^2$	The variance assumed the null hypothesis.
$\sigma1^2$	The variance assumed the alternative hypothesis.
Alpha	The probability of rejecting a true null hypothesis.

Summary Statements

A single-group design (with the mean assumed to be unknown) will be used to test whether the variance (σ^2) is less than 42.5 (H0: $\sigma^2 \geq 42.5$ versus H1: $\sigma^2 < 42.5$). The comparison will be made using a one-sided, one-sample Chi-Square variance-ratio test (with df = N - 1), with a Type I error rate (α) of 0.05. To detect a variance of 29.75 (a variance difference of 12.75) with a sample size of 10 subjects, the power is 0.14448.

Dropout-Inflated Sample Size

Dropout Rate	Sample Size N	Dropout-Inflated Enrollment Sample Size N'	Expected Number of Dropouts D
20%	10	13	3
20%	50	63	13
20%	90	113	23
20%	130	163	33
20%	170	213	43
20%	210	263	53
20%	250	313	63

Dropout Rate	The percentage of subjects (or items) that are expected to be lost at random during the course of the study and for whom no response data will be collected (i.e., will be treated as "missing"). Abbreviated as DR.
N	The evaluable sample size at which power is computed (as entered by the user). If N subjects are evaluated out of the N' subjects that are enrolled in the study, the design will achieve the stated power.
N'	The total number of subjects that should be enrolled in the study in order to obtain N evaluable subjects, based on the assumed dropout rate. N' is calculated by inflating N using the formula N' = N / (1 - DR), with N' always rounded up. (See Julious, S.A. (2010) pages 52-53, or Chow, S.C., Shao, J., Wang, H., and Lokhnygina, Y. (2018) pages 32-33.)
D	The expected number of dropouts. D = N' - N.

Dropout Summary Statements

Anticipating a 20% dropout rate, 13 subjects should be enrolled to obtain a final sample size of 10 subjects.

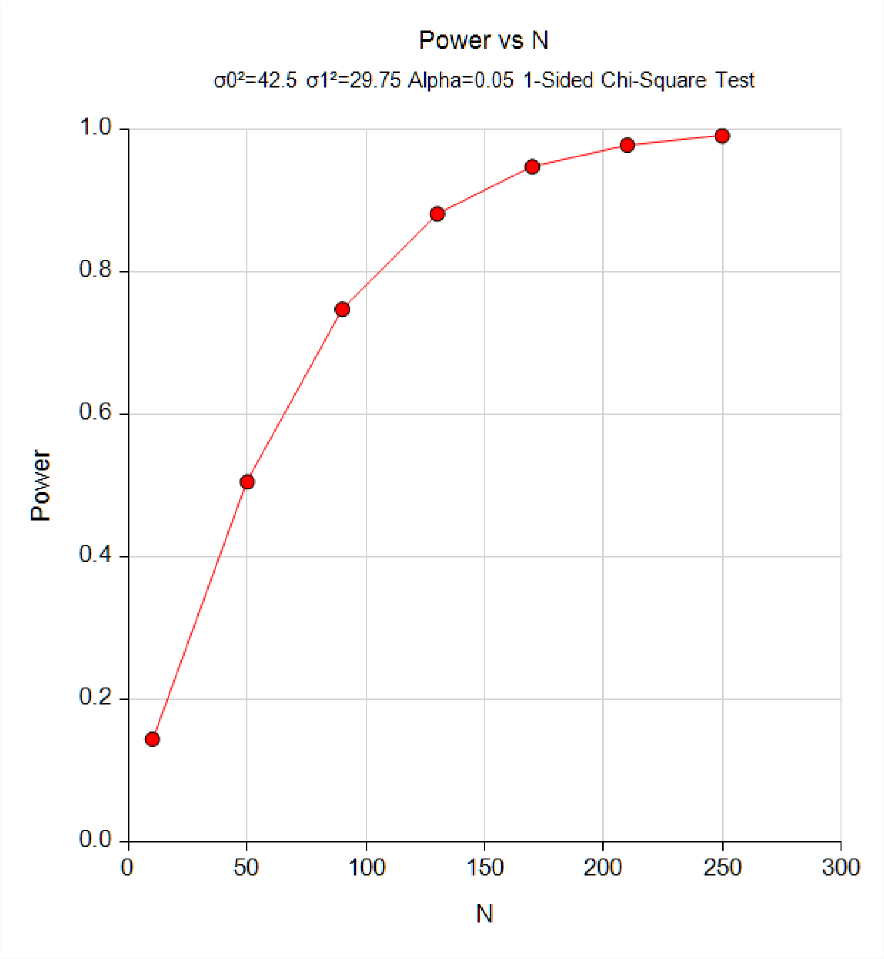
References

Davies, Owen L. 1971. The Design and Analysis of Industrial Experiments. Hafner Publishing Company, New York.
Ostle, B. 1988. Statistics in Research. Fourth Edition. Iowa State Press. Ames, Iowa.
Zar, Jerrold H. 1984. Biostatistical Analysis (Second Edition). Prentice-Hall. Englewood Cliffs, New Jersey.

This report shows the calculated power for each scenario.

Plots Section

Plots



This plot shows the power versus the sample size. We see that a sample size of about 150 is necessary to achieve a power of 0.90.

Example 2 – Calculating Sample Size

Continuing with the previous example, the analyst wants to find the necessary sample sizes to achieve a power of 0.9, for two significance levels, 0.01 and 0.05, and for several variance values. To make interpreting the output easier, the analyst decides to switch from the absolute scale to a ratio scale. To accomplish this, the baseline variance is set at 1.0 and the alternative variances of 0.2, 0.3, 0.4, 0.5, 0.6, and 0.7 are tried.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For .... **Sample Size**
Alternative Hypothesis .... **Ha: σ² < σ0²**
Known Mean .... **Unchecked**
Power **0.90**
Alpha **0.01 0.05**
Input Type .... **Variances**
σ0² (Baseline Variance) **1.0**
σ1² (Alternative Variance) .... **0.2 to 0.7 by 0.1**

Output

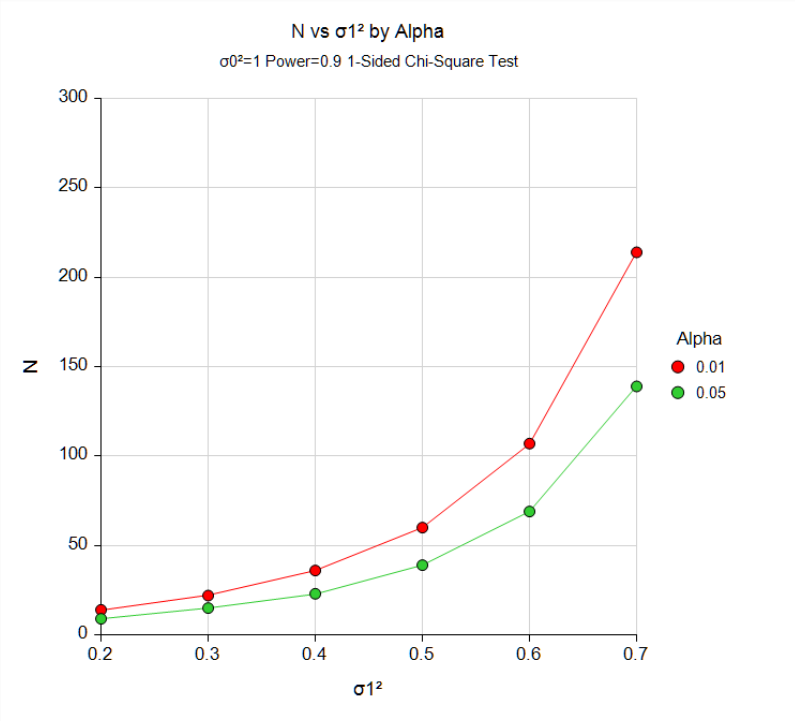
Click the Calculate button to perform the calculations and generate the following output.

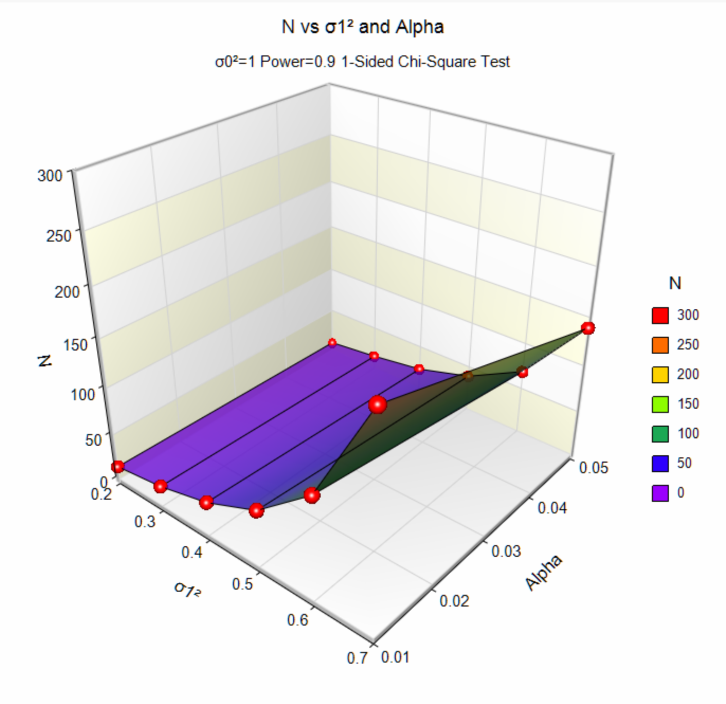
Numeric Results

Solve For: Sample Size
Hypotheses: H0: σ² ≥ σ0² vs. Ha: σ² < σ0²
Mean: Not Known (Chi-Square df = N - 1)

Power	Sample Size N	Variance: Baseline σ0²	Variance: Alternative σ1²	Alpha
0.91734	14	1	0.2	0.01
0.90902	9	1	0.2	0.05
0.90091	22	1	0.3	0.01
0.91935	15	1	0.3	0.05
0.90368	36	1	0.4	0.01
0.90067	23	1	0.4	0.05
0.90163	60	1	0.5	0.01
0.90423	39	1	0.5	0.05
0.90126	107	1	0.6	0.01
0.90078	69	1	0.6	0.05
0.90060	214	1	0.7	0.01
0.90117	139	1	0.7	0.05

Plots





These reports and plots show the necessary sample size for various values of *σ1²*. Note that as *σ1²* gets farther from zero, the required sample size increases.

Example 3 – Validation using Zar (1984)

Zar (1984) page 117 presents an example with $\sigma 0^2$ = 1.5, $\sigma 1^2$ = 2.6898, *N* = 40, *Alpha* = 0.05, and *Power* = 0.84.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 3** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For	**Power**
Alternative Hypothesis	**Ha: $\sigma^2 > \sigma 0^2$**
Known Mean	**Unchecked**
Alpha	**0.05**
N (Sample Size)	**40**
Input Type	**Variances**
$\sigma 0^2$ (Baseline Variance)	**1.5**
$\sigma 1^2$ (Alternative Variance)	**2.6898**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Solve For: Power
Hypotheses: H0: $\sigma^2 \le \sigma 0^2$ vs. Ha: $\sigma^2 > \sigma 0^2$
Mean: Not Known (Chi-Square df = N - 1)

		Variance		
Power	Sample Size N	Baseline $\sigma 0^2$	Alternative $\sigma 1^2$	Alpha
0.83517	40	1.5	2.6898	0.05

PASS calculated the power to be 0.83517, which matches Zar's result of 0.84 within rounding.

Example 4 – Validation using Davies (1971)

Davies (1971) page 40 presents an example of determining *N* when (in the standard deviation scale) $\sigma 0$ = 0.04, $\sigma 1$ = 0.10, Alpha = 0.05, and Power = 0.99. Davies calculates *N* to be 13.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 4** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For	**Sample Size**
Alternative Hypothesis	**Ha: $\sigma^2 > \sigma 0^2$**
Known Mean	**Unchecked**
Power	**0.99**
Alpha	**0.05**
Input Type	**Standard Deviations**
σ0 (Baseline Std. Dev.)	**0.04**
σ1 (Alternative Std. Dev.)	**0.10**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Solve For: Sample Size
Hypotheses: H0: $\sigma^2 \le \sigma 0^2$ vs. Ha: $\sigma^2 > \sigma 0^2$
Mean: Not Known (Chi-Square df = N - 1)

		Standard Deviation		Variance		
Power	Sample Size N	Baseline σ0	Alternative σ1	Baseline $\sigma 0^2$	Alternative $\sigma 1^2$	Alpha
0.99238	13	0.04	0.1	0.002	0.01	0.05

PASS calculated an *N* of 13 which matches Davies' result.