Chapter 435

Tests for Two Exponential Means

Introduction

This program module designs studies for testing hypotheses about the means of two exponential distributions. Such a test is used when you want to make a comparison between two groups that both follow the exponential distribution. The responses from the samples are assumed to be continuous, positive numbers such as lifetime.

We adopt the basic methodology outlined in the books by Bain and Engelhardt (1991) and Desu and Raghavarao (1990).

Technical Details

The test procedure described here makes the assumption that lifetimes in each group follow an exponential distribution. The densities of the two exponential distributions are written as

$$f_i(t) = \frac{1}{\theta_i}\exp\left(-\frac{t}{\theta_i}\right), \quad i = 1,2$$

The parameters θ_i are interpreted as the average failure times, the mean time to failure (MTTF), or the mean time between failures (MTBF) of the two groups. The reliability, or the probability that a unit continues running beyond time *t*, is

$$R_i(t) = e^{-\frac{t}{\theta_i}}$$

Hypothesis Test

The relevant statistical hypothesis is $H_0: \theta_1 / \theta_2 = 1$ versus one of the following alternatives: $H_A: \theta_1 / \theta_2 = \rho > 1$, $H_A: \theta_1 / \theta_2 = \rho < 1$, or $H_A: \theta_1 / \theta_2 = \rho \neq 1$. The test procedure is to reject the null hypothesis H_0 if the ratio of the observed mean lifetimes $\hat{\rho} = \hat{\theta}_1 / \hat{\theta}_2$ is too large or too small. The samples of size n_i are assumed to be drawn without replacement. The experiment is run until all items fail.

If the experiment is curtailed before all $n_1 + n_2$ items fail, the sample size results are based on the number of failures $r_1 + r_2$, not the total number of samples $n_1 + n_2$.

The mean lifetimes are estimated as follows

$$\hat{\theta}_i = \frac{\sum_{\text{over } j} t_{ij}}{r_i}, \quad i = 1,2$$

where t_{ij} is the time that the j^{th} item in the i^{th} group is tested, whether measured until failure or until the study is completed.

Power and sample size calculations are based on the fact that the estimated lifetime ratio is proportional to the *F* distribution. That is,

$$\frac{\hat{\theta}_1}{\hat{\theta}_2} \sim \frac{\theta_1}{\theta_2} F_{r_1, r_2}$$

which, under the null hypothesis of equality, becomes

$$\frac{\hat{\theta}_1}{\hat{\theta}_2} \sim F_{r_1, r_2}$$

Note that only the actual numbers of failures are used in these distributions. Hence, we assume that the experiment is run until all items fail so that $r_i = n_i$. That is, the sample sizes are the number of failures, not the number of items. Enough units must be sampled to ensure that the stated number of failures occur.

Example 1 – Power for Several Sample Sizes

This example will calculate power for several sample sizes of a study designed to compare the average failure time of (supposedly) identical components manufactured by two companies. Management wants the study to be large enough to detect a ratio of mean lifetimes of 1.3 at the 0.05 significance level. The analysts decide to look at sample sizes between 5 and 500.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For	**Power**
Alternative Hypothesis	**Ha: Θ1 ≠ Θ2**
Alpha	**0.05**
Group Allocation	**Equal (N1 = N2)**
Sample Size Per Group	**5 20 50 100 200 300 400 500**
Θ1 (Group 1 Mean Life)	**1.3**
Θ2 (Group 2 Mean Life)	**1.0**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Reports

Numeric Results

Solve For: Power
Hypotheses: H0: Θ1 = Θ2 vs. Ha: Θ1 ≠ Θ2

	Sample Size			Mean Life			
Power	N1	N2	N	Θ1	Θ2	Ratio Θ1 / Θ2	Alpha
0.06652	5	5	10	1.3	1	1.3	0.05
0.12839	20	20	40	1.3	1	1.3	0.05
0.25602	50	50	100	1.3	1	1.3	0.05
0.45619	100	100	200	1.3	1	1.3	0.05
0.74551	200	200	400	1.3	1	1.3	0.05
0.89447	300	300	600	1.3	1	1.3	0.05
0.95976	400	400	800	1.3	1	1.3	0.05
0.98559	500	500	1000	1.3	1	1.3	0.05

Power	The probability of rejecting a false null hypothesis when the alternative hypothesis is true.
N1 and N2	The number of failures needed in Groups 1 and 2.
N	The total sample size. N = N1 + N2.
Θ1 and Θ2	The Mean Life in Groups 1 and 2 at which power and sample size calculations are made.
Θ1 / Θ2	The simple ratio of Θ1 to Θ2.
Alpha	The probability of rejecting a true null hypothesis.

Summary Statements

A parallel two-group design will be used to test whether the Group 1 Exponential mean (θ1) is different from the Group 2 Exponential mean (θ2) (H0: θ1 = θ2 versus Ha: θ1 ≠ θ2). The comparison will be made using a two-sided, two-sample F-distribution test, with a Type I error rate (α) of 0.05. To detect Exponential (lifetime) means of 1.3 for Group 1 and 1 for Group 2, with a sample size of 5 in Group 1 and 5 in Group 2, the power is 0.06652.

References

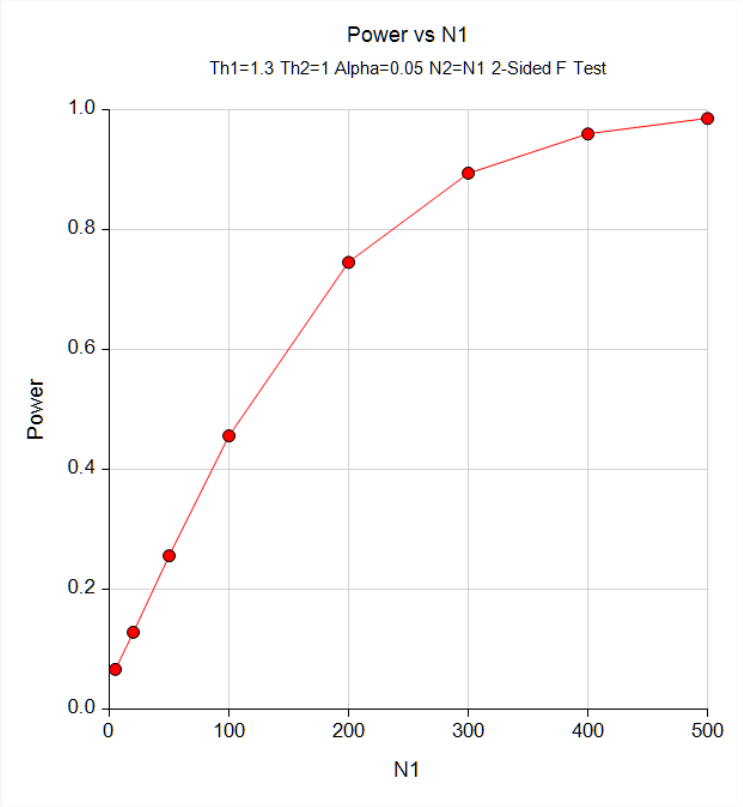
Bain, L.J. and Engelhardt, M. 1991. Statistical Analysis of Reliability and Life-Testing Models. Marcel Dekker. New York.

Desu, M. M. and Raghavarao, D. 1990. Sample Size Methodology. Academic Press. New York.

This report shows the power for each of the scenarios.

Plots Section

Plots



This plot shows the relationship between power and sample size.

Example 2 – Validation using Manual Calculations

We could not find published results that could be used to validate this procedure. Instead, we will compare the results to those computed using our probability distribution calculator.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab	
Solve For	**Power**
Alternative Hypothesis	**Ha: Θ1 > Θ2**
Alpha	**0.05**
Group Allocation	**Equal (N1 = N2)**
Sample Size Per Group	**20**
Θ1 (Group 1 Mean Life)	**1.3**
Θ2 (Group 2 Mean Life)	**1.0**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Solve For: Power
Hypotheses: H0: Θ1 = Θ2 vs. Ha: Θ1 > Θ2

	Sample Size			Mean Life			
Power	N1	N2	N	Θ1	Θ2	Ratio Θ1 / Θ2	Alpha
0.20369	20	20	40	1.3	1	1.3	0.05

We will now check these results using manual calculations. First, we find critical value

$$F_{0.95,40,40} = 1.6927972097$$

using the probability calculator. Now, to calculate the power, we find the inverse *F* of 1.6927972097/1.3 = 1.302152 to be 0.79631. One minus 0.79631 is 0.20369, which matches the reported value of Power.