Chapter 435

Tests for Two Exponential Means

Introduction

This program module designs studies for testing hypotheses about the means of two exponential distributions. Such a test is used when you want to make a comparison between two groups that both follow the exponential distribution. The responses from the samples are assumed to be continuous, positive numbers such as lifetime.

We adopt the basic methodology outlined in the books by Bain and Engelhardt (1991) and Desu and Raghavarao (1990).

Technical Details

The test procedure described here makes the assumption that lifetimes in each group follow an exponential distribution. The densities of the two exponential distributions are written as

$$f_i(t) = \frac{1}{\theta_i} \exp\left(-\frac{t}{\theta_i}\right), \quad i = 1,2$$

The parameters θ_i are interpreted as the average failure times, the mean time to failure (MTTF), or the mean time between failures (MTBF) of the two groups. The reliability, or the probability that a unit continues running beyond time t, is

$$R_i(t) = e^{-\frac{t}{\theta_i}}$$

Hypothesis Test

The relevant statistical hypothesis is $H_0: \theta_1 / \theta_2 = 1$ versus one of the following alternatives: $H_A: \theta_1 / \theta_2 = \rho > 1$, $H_A: \theta_1 / \theta_2 = \rho < 1$, or $H_A: \theta_1 / \theta_2 = \rho \neq 1$. The test procedure is to reject the null hypothesis H_0 if the ratio of the observed mean lifetimes $\hat{\rho} = \hat{\theta}_1 / \hat{\theta}_2$ is too large or too small. The samples of size n_i are assumed to be drawn without replacement. The experiment is run until all items fail.

If the experiment is curtailed before all $n_1 + n_2$ items fail, the sample size results are based on the number of failures $r_1 + r_2$, not the total number of samples $n_1 + n_2$.

The mean lifetimes are estimated as follows

$$\hat{\theta}_i = \frac{\sum_{\text{over } j} t_{ij}}{r_i}, \quad i = 1,2$$

where t_{ij} is the time that the jth item in the ith group is tested, whether measured until failure or until the study is completed.

Power and sample size calculations are based on the fact that the estimated lifetime ratio is proportional to the *F* distribution. That is,

$$\frac{\hat{\theta}_1}{\hat{\theta}_2} \sim \frac{\theta_1}{\theta_2} F_{r_1, r_2}$$

which, under the null hypothesis of equality, becomes

$$\frac{\hat{\theta}_1}{\hat{\theta}_2} \sim F_{r_1, r_2}$$

Note that only the actual numbers of failures are used in these distributions. Hence, we assume that the experiment is run until all items fail so that $r_i = n_i$. That is, the sample sizes are the number of failures, not the number of items. Enough units must be sampled to ensure that the stated number of failures occur.

Procedure Options

This section describes the options that are specific to this procedure. These are located on the Design tab. For more information about the options of other tabs, go to the Procedure Window chapter.

Design Tab

The Design tab contains most of the parameters and options that you will be concerned with.

Solve For

Solve For

This option specifies the parameter to be solved for from the other parameters. Under most situations, you will select either *Power* or *Sample Size*.

Select *Sample Size* when you want to calculate the sample size needed to achieve a given power and alpha level.

Select *Power* when you want to calculate the power of an experiment.

Test

Alternative Hypothesis

Specify the alternative hypothesis of the test. Since the null hypothesis is equality (a difference between theta1 and theta2 of zero), the alternative is all that needs to be specified.

Note that the alternative hypothesis should match the values of Theta1 and Theta2. That is, if you select Ha: Theta1 > Theta, then the value of Theta1 should be greater than the value of Theta2.

Power and Alpha

Power = 1–Beta (Beta is Consumer's Risk)

This option specifies one or more values for power. Power is the probability of rejecting a false null hypothesis, and is equal to one minus Beta. Beta (consumer's risk) is the probability of a type-II error, which occurs when a false null hypothesis is not rejected. In this procedure, a type-II error occurs when you fail to reject the null hypothesis of equal thetas when in fact they are different.

Values must be between zero and one. Historically, the value of 0.80 (Beta = 0.20) was used for power. Now, 0.90 (Beta = 0.10) is also commonly used.

A single value may be entered here or a range of values such as *0.8 to 0.95 by 0.05* may be entered.

Alpha

This option specifies one or more values for the probability of a type-I error. A type-I error occurs when a true null hypothesis is rejected. In this procedure, a type-I error occurs when you reject the null hypothesis of equal thetas when in fact they are equal.

Values must be between zero and one. Historically, the value of 0.05 has been used for alpha. This means that about one test in twenty will falsely reject the null hypothesis. You should pick a value for alpha that represents the risk of a type-I error you are willing to take in your experimental situation.

You may enter a range of values such as *0.01 0.05 0.10* or *0.01 to 0.10 by 0.01*.

Sample Size (When Solving for Sample Size)

Group Allocation

Select the option that describes the constraints on *N1* or *N2* or both.

The options are

- **Equal (N1 = N2)**

  This selection is used when you wish to have equal sample sizes in each group. Since you are solving for both sample sizes at once, no additional sample size parameters need to be entered.

- **Enter N1, solve for N2**

  Select this option when you wish to fix *N1* at some value (or values), and then solve only for *N2*. Please note that for some values of *N1*, there may not be a value of *N2* that is large enough to obtain the desired power.

- **Enter N2, solve for N1**

  Select this option when you wish to fix *N2* at some value (or values), and then solve only for *N1*. Please note that for some values of *N2*, there may not be a value of *N1* that is large enough to obtain the desired power.

- **Enter R = N2/N1, solve for N1 and N2**

  For this choice, you set a value for the ratio of *N2* to *N1*, and then PASS determines the needed *N1* and *N2*, with this ratio, to obtain the desired power. An equivalent representation of the ratio, *R*, is

  $$N2 = R * N1.$$

- **Enter percentage in Group 1, solve for N1 and N2**

  For this choice, you set a value for the percentage of the total sample size that is in Group 1, and then PASS determines the needed *N1* and *N2* with this percentage to obtain the desired power.

N1 (Sample Size, Group 1)

This option is displayed if Group Allocation = "Enter N1, solve for N2"

N1 is the number of items or individuals sampled from the Group 1 population.

N1 must be ≥ 2. You can enter a single value or a series of values.

N2 (Sample Size, Group 2)

This option is displayed if Group Allocation = "Enter N2, solve for N1"

N2 is the number of items or individuals sampled from the Group 2 population.

N2 must be ≥ 2. You can enter a single value or a series of values.

R (Group Sample Size Ratio)

This option is displayed only if Group Allocation = "Enter R = N2/N1, solve for N1 and N2."

R is the ratio of *N2* to *N1*. That is,

$$R = N2 / N1.$$

Use this value to fix the ratio of *N2* to *N1* while solving for *N1* and *N2*. Only sample size combinations with this ratio are considered.

N2 is related to *N1* by the formula:

$$N2 = [R \times N1],$$

where the value *[Y]* is the next integer ≥ *Y*.

For example, setting *R* = 2.0 results in a Group 2 sample size that is double the sample size in Group 1 (e.g., *N1* = 10 and *N2* = 20, or *N1* = 50 and *N2* = 100).

R must be greater than 0. If *R* < 1, then *N2* will be less than *N1*; if *R* > 1, then *N2* will be greater than *N1*. You can enter a single or a series of values.

Percent in Group 1

This option is displayed only if Group Allocation = "Enter percentage in Group 1, solve for N1 and N2."

Use this value to fix the percentage of the total sample size allocated to Group 1 while solving for *N1* and *N2*. Only sample size combinations with this Group 1 percentage are considered. Small variations from the specified percentage may occur due to the discrete nature of sample sizes.

The Percent in Group 1 must be greater than 0 and less than 100. You can enter a single or a series of values.

Sample Size (When <u>Not</u> Solving for Sample Size)

Group Allocation

Select the option that describes how individuals in the study will be allocated to Group 1 and to Group 2.

The options are

- **Equal (N1 = N2)**

  This selection is used when you wish to have equal sample sizes in each group. A single per group sample size will be entered.

- **Enter N1 and N2 individually**

  This choice permits you to enter different values for *N1* and *N2*.

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- **Enter N1 and R, where N2 = R * N1**

  Choose this option to specify a value (or values) for *N1*, and obtain *N2* as a ratio (multiple) of *N1*.

- **Enter total sample size and percentage in Group 1**

  Choose this option to specify a value (or values) for the total sample size (*N*), obtain *N1* as a percentage of *N*, and then *N2* as *N - N1*.

Sample Size Per Group

This option is displayed only if Group Allocation = "Equal (N1 = N2)."

The Sample Size Per Group is the number of items or individuals sampled from each of the Group 1 and Group 2 populations. Since the sample sizes are the same in each group, this value is the value for *N1*, and also the value for *N2*.

The Sample Size Per Group must be ≥ 2. You can enter a single value or a series of values.

N1 (Sample Size, Group 1)

This option is displayed if Group Allocation = "Enter N1 and N2 individually" or "Enter N1 and R, where N2 = R * N1."

N1 is the number of items or individuals sampled from the Group 1 population.

N1 must be ≥ 2. You can enter a single value or a series of values.

N2 (Sample Size, Group 2)

This option is displayed only if Group Allocation = "Enter N1 and N2 individually."

N2 is the number of items or individuals sampled from the Group 2 population.

N2 must be ≥ 2. You can enter a single value or a series of values.

R (Group Sample Size Ratio)

This option is displayed only if Group Allocation = "Enter N1 and R, where N2 = R * N1."

R is the ratio of *N2* to *N1*. That is,

$$R = N2/N1$$

Use this value to obtain *N2* as a multiple (or proportion) of *N1*.

N2 is calculated from *N1* using the formula:

$$N2=[R \times N1],$$

where the value *[Y]* is the next integer ≥ *Y*.

For example, setting *R* = 2.0 results in a Group 2 sample size that is double the sample size in Group 1.

R must be greater than 0. If *R* < 1, then *N2* will be less than N1; if *R* > 1, then *N2* will be greater than *N1*. You can enter a single value or a series of values.

Total Sample Size (N)

This option is displayed only if Group Allocation = "Enter total sample size and percentage in Group 1."

This is the total sample size, or the sum of the two group sample sizes. This value, along with the percentage of the total sample size in Group 1, implicitly defines *N1* and *N2*.

The total sample size must be greater than one, but practically, must be greater than 3, since each group sample size needs to be at least 2.

You can enter a single value or a series of values.

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Percent in Group 1

This option is displayed only if Group Allocation = "Enter total sample size and percentage in Group 1."

This value fixes the percentage of the total sample size allocated to Group 1. Small variations from the specified percentage may occur due to the discrete nature of sample sizes.

The Percent in Group 1 must be greater than 0 and less than 100. You can enter a single value or a series of values.

Effect Size

Theta1 (Group 1 Mean Life)

Enter one or more values for the *mean life* of group 1 under the alternative hypothesis. This value is usually scaled in terms of elapsed time such as hours, days, or years. Of course, all time values must be on the same time scale.

Note that the value of theta may be calculated from the estimated probability of failure using the relationship

$$P(Failure) = 1 - e^{-time/\theta}$$

so that

$$\theta = \frac{-time}{\ln(1 - P(Failure))}$$

Any positive values are valid. You may enter a range of values such as '10 20 30' or '100 to 1000 by 100.'

Note that only the ratio of theta1 and theta2 is used in the calculations.

Theta2 (Group 2 Mean Life)

Enter one or more values for the *mean life* of group 2 under the alternative hypothesis. This value is usually scaled in terms of elapsed time such as hours, days, or years. Of course, all time values must be on the same time scale.

Note that the value of theta may be calculated from the estimated probability of failure using the relationship

$$P(Failure) = 1 - e^{-time/\theta}$$

so that

$$\theta = \frac{-time}{\ln(1 - P(Failure))}$$

Any positive values are valid. You may enter a range of values such as '10 20 30' or '100 to 1000 by 100.'

Note that only the ratio of theta1 and theta2 is used in the calculations.

Example 1 – Power for Several Sample Sizes

This example will calculate power for several sample sizes of a study designed to compare the average failure time of (supposedly) identical components manufactured by two companies. Management wants the study to be large enough to detect a ratio of mean lifetimes of 1.3 at the 0.05 significance level. The analysts decide to look at sample sizes between 5 and 500.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Tests for Two Exponential Means** procedure window by expanding **Means**, then **Two Independent Means**, then clicking on **Non-Normal Data**, and then clicking on **Tests for Two Exponential Means**. You may then make the appropriate entries as listed below, or open **Example 1** by going to the **File** menu and choosing **Open Example Template**.

Option	Value
Design Tab	
Solve For	**Power**
Alternative Hypothesis	**Ha: Theta1 ≠ Theta2**
Alpha..	**0.05**
Group Allocation	**Equal (N1 = N2)**
Sample Size Per Group	**5 20 50 100 200 300 400 500**
Theta1 (Group 1 Mean Life)	**1.3**
Theta2 (Group 2 Mean Life)	**1.0**

Annotated Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Numeric Results
H0: Theta1 = Theta2. Ha: Theta1 ≠ Theta2.

Power	N1	N2	N	Theta1	Theta2	Theta1/ Theta2	Alpha
0.06652	5	5	10	1.3	1.0	1.300	0.050
0.12839	20	20	40	1.3	1.0	1.300	0.050
0.25602	50	50	100	1.3	1.0	1.300	0.050
0.45619	100	100	200	1.3	1.0	1.300	0.050
0.74551	200	200	400	1.3	1.0	1.300	0.050
0.89447	300	300	600	1.3	1.0	1.300	0.050
0.95976	400	400	800	1.3	1.0	1.300	0.050
0.98559	500	500	1000	1.3	1.0	1.300	0.050

Report Definitions
Power is the probability of rejecting a false null hypothesis.
N1 and N2 are the number of failures needed in Groups 1 and 2.
N is the total sample size, N1 + N2.
Theta1 and Theta2 are the Mean Lives in Groups 1 and 2 at which power and sample size calculations are made.
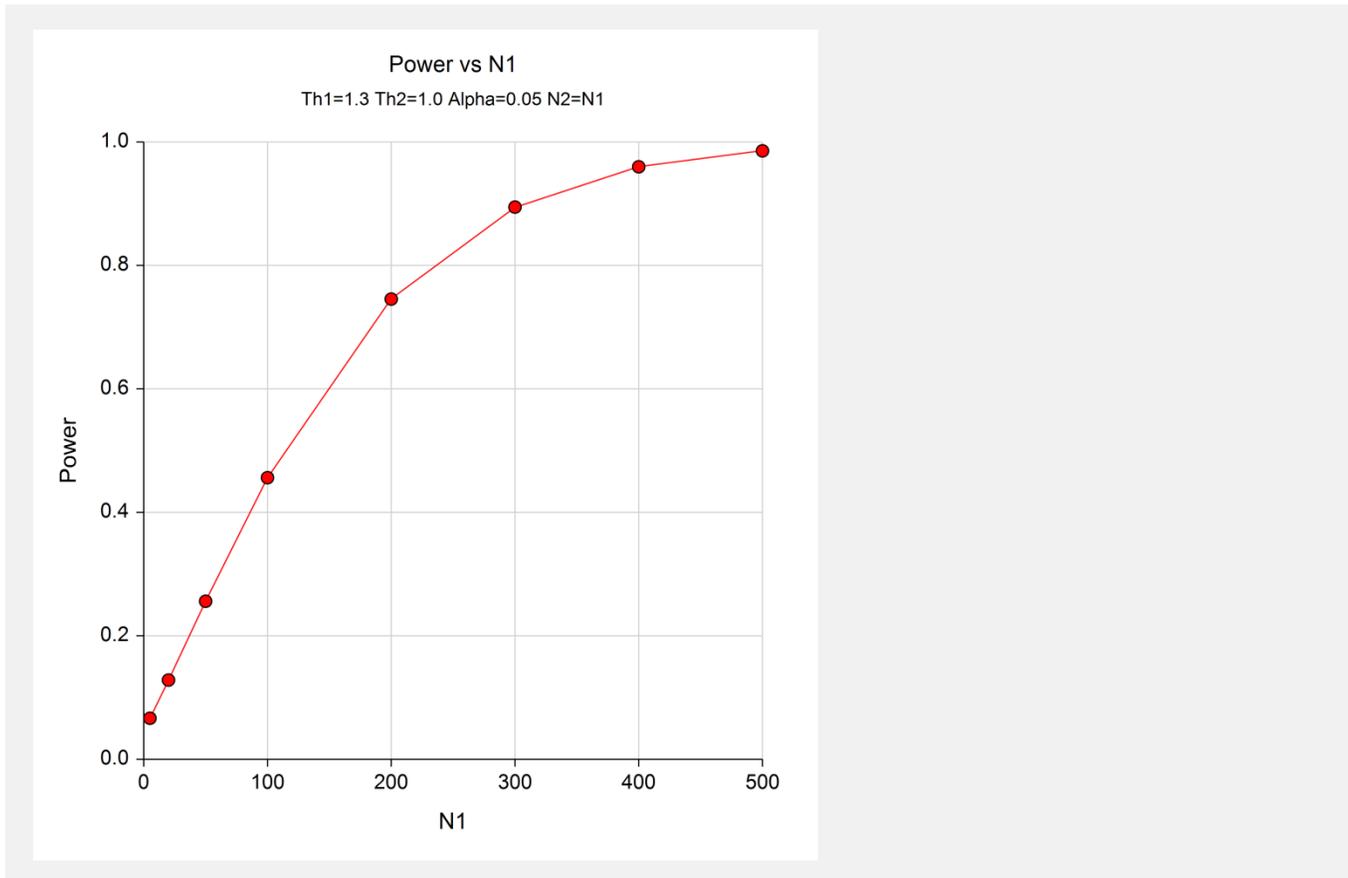Theta1 / Theta2 is the simple ratio of Theta1 to Theta2.
Alpha is the probability of rejecting a true null hypothesis.

This report shows the power for each of the scenarios.

Plots Section

This plot shows the relationship between power and sample size.

Example 2 – Validation using Manual Calculations

We could not find published results that could be used to validate this procedure. Instead, we will compare the results to those computed using our probability distribution calculator.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Tests for Two Exponential Means** procedure window by expanding **Means**, then **Two Independent Means**, then clicking on **Non-Normal Data**, and then clicking on **Tests for Two Exponential Means**. You may then make the appropriate entries as listed below, or open **Example 2** by going to the **File** menu and choosing **Open Example Template**.

Option	**Value**
Design Tab	
Solve For	**Power**
Alternative Hypothesis	**Ha: Theta1 > Theta2**
Alpha..	**0.05**
Group Allocation	**Equal (N1 = N2)**
Sample Size Per Group	**20**
Theta1 (Group 1 Mean Life)	**1.3**
Theta2 (Group 2 Mean Life)	**1.0**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Numeric Results
H0: Theta1 = Theta2. Ha: Theta1 > Theta2.

Power	N1	N2	N	Theta1	Theta2	Theta1/ Theta2	Alpha
0.20369	20	20	40	1.3	1.0	1.300	0.050

We will now check these results using manual calculations. First, we find critical value

$$F_{0.95,40,40} = 1.6927972097$$

using the probability calculator. Now, to calculate the power, we find the inverse F of $1.6927972097/1.3 = 1.302152$ to be 0.79631. One minus 0.79631 is 0.20369, which matches the reported value of Power.