Chapter 436

Tests for the Difference Between Two Poisson Rates

Introduction

The Poisson probability law gives the probability distribution of the number of events occurring in a specified interval of time or space. The Poisson distribution is often used to fit count data, such as the number of defects on an item, the number of accidents at an intersection during a year, the number of calls to a call center during an hour, or the number of meteors seen in the evening sky during an hour.

The Poisson distribution is characterized by a single parameter which is the mean number of occurrences during the specified interval.

The procedure documented in this chapter calculates the power or sample size for testing whether the difference of two Poisson rates is different from zero.

Technical Details

These results follow Mathews (2010). Assume that all subjects in each group are observed for a fixed time period (which we set as one without loss of generality and the number of events (outcomes or defects) is recorded. The following table presents the various terms used.

Group	1	2
Sample Size	N_1	N_2
Number of Events	X_1	X_2
Event Rate	λ_1	λ_2
Distribution of X	Poisson(λ_1)	Poisson(λ_2)

Mathews (2010) proposed two test statistics that can be used to test statistical hypotheses about the rate difference. The first is based is the *large-sample z-test* of the hypotheses H0: $\lambda_1 = \lambda_2$ versus Ha: $\lambda_1 \neq \lambda_2$.

$$z_{LS} = \frac{\hat{\lambda}_2 - \hat{\lambda}_1}{\sqrt{\dfrac{\hat{\lambda}_1}{N_1} + \dfrac{\hat{\lambda}_2}{N_2}}}$$

where $\hat{\lambda}_i = X_i/N_i$. This test is appropriate when $X_i > 30$.

The power of the two-sided level α hypothesis test is given by

$$Power = \Phi\left[\frac{\lambda_2 - \lambda_1}{\sqrt{\frac{\lambda_1}{N_1} + \frac{\lambda_2}{N_2}}} - z_{1-\alpha/2}\right]$$

where $z_x = \Phi(x)$ is the standard normal distribution function.

The second test is the *square root transform* version of the first. The square root transform helps the test statistic approach normality for smaller sample sizes. It is given by

$$z_{SR} = \frac{\sqrt{\hat{\lambda}_2} - \sqrt{\hat{\lambda}_1}}{\frac{1}{2}\sqrt{\frac{1}{N_1} + \frac{1}{N_2}}}$$

The power of the two-sided level α hypothesis test is given by

$$Power = \Phi\left[\frac{\sqrt{\lambda_2} - \sqrt{\lambda_1}}{\frac{1}{2}\sqrt{\frac{1}{N_1} + \frac{1}{N_2}}} - z_{1-\alpha/2}\right]$$

Procedure Options

This section describes the options that are specific to this procedure. These are located on the Design tab. For more information about the options of other tabs, go to the Procedure Window chapter.

Design Tab

The Design tab contains most of the parameters and options that you will be concerned with.

Solve For

Solve For

This option specifies the parameter to be solved for from the other parameters. The parameters that may be selected are $\lambda 2$, *Sample Size*, and *Power*.

Note that the value selected here always appears as the vertical axis on the charts.

The program is set up to calculate power directly. To find appropriate values of the other parameters, a binary search is made using an iterative procedure until an appropriate value is found.

Test

Alternative Hypothesis

Specify whether the alternative hypothesis of the test is one-sided or two-sided. If a one-sided test is chosen, the hypothesis test direction is chosen based on whether the event rate difference ($\lambda 2-\lambda 1$) is greater than or less than 0.

The options are

- **Two-Sided**

  The null and alternative hypotheses are

  $H_0: \lambda_2 - \lambda_1 = 0$ vs. $H_A: \lambda_2 - \lambda_1 \neq 0$

- **One-Sided**

  The upper and lower null and alternative hypotheses are

  Upper: $H_0: \lambda_2 - \lambda_1 \leq 0$ vs. $H_A: \lambda_2 - \lambda_1 > 0$

  Lower: $H_0: \lambda_2 - \lambda_1 \geq 0$ vs. $H_A: \lambda_2 - \lambda_1 < 0$

Test Statistic

Two test statistics are available to test whether the difference between two event rates is significantly different from zero: Large Samples and Square-Root Transform.

- **Large Samples**

  Large samples are designated as those in which N1 x λ1 > 30 and N2 x λ2 > 30. When this sample size requirement is met, the distribution of the difference of the sample event rates as approximately normal and so it can be used as a test.

- **Square-Root Transform**

  If the sample size requirements above are not met, the difference of the square roots of the sample event rates tends toward normality more quickly. Thus, it can be used as a test.

Power and Alpha

Power

This option specifies one or more values for power. Power is the probability of rejecting a false null hypothesis, and is equal to one minus beta. Beta is the probability of a type-II error, which occurs when a false null hypothesis is not rejected.

Values must be between zero and one. Historically, the value of 0.80 (beta = 0.20) was used for power. Now, 0.90 (beta = 0.10) is commonly used.

A single value may be entered or a range of values, such as *0.8 to 0.95 by 0.05,* may be entered.

Alpha

This option specifies one or more values for the probability of a type-I error. A type-I error occurs when a true null hypothesis is rejected.

Values must be between zero and one. Usually, the value of 0.05 is used for two-sided tests and 0.025 is used for one-sided tests.

You may enter a range of values such as *0.01 0.05 0.10* or *0.01 to 0.10 by 0.01.*

Sample Size (When Solving for Sample Size)

Group Allocation

Select the option that describes the constraints on *N1* or *N2* or both.

The options are

- **Equal (N1 = N2)**

  This selection is used when you wish to have equal sample sizes in each group. Since you are solving for both sample sizes at once, no additional sample size parameters need to be entered.

- **Enter N1, solve for N2**

  Select this option when you wish to fix *N1* at some value (or values), and then solve only for *N2*. Please note that for some values of *N1*, there may not be a value of *N2* that is large enough to obtain the desired power.

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- **Enter N2, solve for N1**

  Select this option when you wish to fix *N2* at some value (or values), and then solve only for *N1*. Please note that for some values of *N2*, there may not be a value of *N1* that is large enough to obtain the desired power.

- **Enter R = N2/N1, solve for N1 and N2**

  For this choice, you set a value for the ratio of *N2* to *N1*, and then PASS determines the needed *N1* and *N2*, with this ratio, to obtain the desired power. An equivalent representation of the ratio, *R*, is

  $$N2 = R * N1.$$

- **Enter percentage in Group 1, solve for N1 and N2**

  For this choice, you set a value for the percentage of the total sample size that is in Group 1, and then PASS determines the needed *N1* and *N2* with this percentage to obtain the desired power.

N1 (Sample Size, Group 1)

This option is displayed if Group Allocation = "Enter N1, solve for N2"

N1 is the number of items or individuals sampled from the Group 1 population.

N1 must be ≥ 2. You can enter a single value or a series of values.

N2 (Sample Size, Group 2)

This option is displayed if Group Allocation = "Enter N2, solve for N1"

N2 is the number of items or individuals sampled from the Group 2 population.

N2 must be ≥ 2. You can enter a single value or a series of values.

R (Group Sample Size Ratio)

This option is displayed only if Group Allocation = "Enter R = N2/N1, solve for N1 and N2."

R is the ratio of *N2* to *N1*. That is,

$$R = N2 / N1.$$

Use this value to fix the ratio of *N2* to *N1* while solving for *N1* and *N2*. Only sample size combinations with this ratio are considered.

N2 is related to *N1* by the formula:

$$N2 = [R \times N1],$$

where the value *[Y]* is the next integer $\geq Y$.

For example, setting *R* = 2.0 results in a Group 2 sample size that is double the sample size in Group 1 (e.g., *N1* = 10 and *N2* = 20, or *N1* = 50 and *N2* = 100).

R must be greater than 0. If *R* < 1, then *N2* will be less than *N1*; if *R* > 1, then *N2* will be greater than *N1*. You can enter a single or a series of values.

Percent in Group 1

This option is displayed only if Group Allocation = "Enter percentage in Group 1, solve for N1 and N2."

Use this value to fix the percentage of the total sample size allocated to Group 1 while solving for *N1* and *N2*. Only sample size combinations with this Group 1 percentage are considered. Small variations from the specified percentage may occur due to the discrete nature of sample sizes.

The Percent in Group 1 must be greater than 0 and less than 100. You can enter a single or a series of values.

Sample Size (When <u>Not</u> Solving for Sample Size)

Group Allocation

Select the option that describes how individuals in the study will be allocated to Group 1 and to Group 2.

The options are

- **Equal (N1 = N2)**

  This selection is used when you wish to have equal sample sizes in each group. A single per group sample size will be entered.

- **Enter N1 and N2 individually**

  This choice permits you to enter different values for *N1* and *N2*.

- **Enter N1 and R, where N2 = R * N1**

  Choose this option to specify a value (or values) for *N1*, and obtain *N2* as a ratio (multiple) of *N1*.

- **Enter total sample size and percentage in Group 1**

  Choose this option to specify a value (or values) for the total sample size (*N*), obtain *N1* as a percentage of *N*, and then *N2* as *N - N1*.

Sample Size Per Group

This option is displayed only if Group Allocation = "Equal (N1 = N2)."

The Sample Size Per Group is the number of items or individuals sampled from each of the Group 1 and Group 2 populations. Since the sample sizes are the same in each group, this value is the value for *N1*, and also the value for *N2*.

The Sample Size Per Group must be ≥ 2. You can enter a single value or a series of values.

N1 (Sample Size, Group 1)

*This option is displayed if Group Allocation = "Enter N1 and N2 individually" or "Enter N1 and R, where N2 = R * N1."*

N1 is the number of items or individuals sampled from the Group 1 population.

N1 must be ≥ 2. You can enter a single value or a series of values.

N2 (Sample Size, Group 2)

This option is displayed only if Group Allocation = "Enter N1 and N2 individually."

N2 is the number of items or individuals sampled from the Group 2 population.

N2 must be ≥ 2. You can enter a single value or a series of values.

R (Group Sample Size Ratio)

*This option is displayed only if Group Allocation = "Enter N1 and R, where N2 = R * N1."*

R is the ratio of *N2* to *N1*. That is,

$$R = N2/N1$$

Use this value to obtain *N2* as a multiple (or proportion) of *N1*.

N2 is calculated from *N1* using the formula:

$$N2=[R \ x \ N1],$$

where the value *[Y]* is the next integer ≥ *Y*.

For example, setting $R = 2.0$ results in a Group 2 sample size that is double the sample size in Group 1.

R must be greater than 0. If $R < 1$, then $N2$ will be less than N1; if $R > 1$, then $N2$ will be greater than $N1$. You can enter a single value or a series of values.

Total Sample Size (N)

This option is displayed only if Group Allocation = "Enter total sample size and percentage in Group 1."

This is the total sample size, or the sum of the two group sample sizes. This value, along with the percentage of the total sample size in Group 1, implicitly defines $N1$ and $N2$.

The total sample size must be greater than one, but practically, must be greater than 3, since each group sample size needs to be at least 2.

You can enter a single value or a series of values.

Percent in Group 1

This option is displayed only if Group Allocation = "Enter total sample size and percentage in Group 1."

This value fixes the percentage of the total sample size allocated to Group 1. Small variations from the specified percentage may occur due to the discrete nature of sample sizes.

The Percent in Group 1 must be greater than 0 and less than 100. You can enter a single value or a series of values.

Effect Size

λ1 (Event Rate of Group 1)

Enter a value (or range of values) for the mean event rate per unit time in group 1 (control group). The value must be greater than zero. This value is compared to λ2 by the statistical test. The difference in the rates, λ2 - λ1, is the amount that this design can detect.

Enter a value (or range of values) for the mean event rate per time unit in the control group (group 1).

Example of Estimating λ1

If 200 patients were exposed for 1 year and 40 experienced the event of interest, then the mean event rate would be

  $λ1 = 40/(200*1) = 0.2$ per patient-year

If 200 patients were exposed for 2 years and 40 experienced the event of interest, then the mean event rate would be

  $λ1 = 40/(200*2) = 0.1$ per patient-year

Event Rate Difference

λ1 is used with λ2 to calculate the event rate difference as

$$Diff = λ2 - λ1$$

such that

$$λ1 = λ2 - Diff$$

The range of acceptable values is $λ1 > 0$. You can enter a single value such as *1* or a series of values such as *1 to 2 by 0.5*.

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Enter λ2, Diff, or Ratio for Group 2

This option lets you indicate how you want to enter λ2. The options are

- **λ2 (Event Rate of Group 2)**

  Enter the value of λ2 directly.

- **Diff (Difference Between Event Rates)**

  Enter values for the difference between the event rates (Diff = λ2 - λ1). The value of λ2 is equal to λ1 + Diff.

- **RR (Ratio of Event Rates)**

  Enter values for the ratio of the event rates (RR = λ2/λ1). The value of λ2 is equal to λ1 x Ratio. Note that the hypothesis still concerns the difference. This is just a convenient way of specifying a value.

λ2 (Event Rate of Group 2)

This option is displayed only if Enter λ2, Diff, or Ratio for Group 2 = "λ2 (Event Rate of Group 2)."

Enter a value (or range of values) for the mean event rate per time unit in group 2 (treatment group). The value must be greater than zero and different from λ1. This value is compared to λ1 by the statistical test. The difference in the rates, λ2 - λ1, is the amount that this design can detect.

Example of Estimating λ2

If 200 patients were exposed for 1 year and 40 experienced the event of interest, then the mean event rate would be

$$\lambda2 = 40/(200*1) = 0.2 \text{ per patient-year}$$

If 200 patients were exposed for 2 years and 40 experienced the event of interest, then the mean event rate would be

$$\lambda2 = 40/(200*2) = 0.1 \text{ per patient-year}$$

Event Rate Difference

λ2 is used with λ1 to calculate the event rate difference as

$$Diff = \lambda2 - \lambda1$$

such that

$$\lambda2 = \lambda1 + Diff$$

The range of acceptable values is λ2 > 0. You can enter a single value such as *1* or a series of values such as *1 to 2 by 0.5*.

λ2 (Event Rate of Group 2) [when solving for λ2 (Event Rate of Group 2)]

If you are solving for λ2, this option will let you indicate whether you want to search less than or greater than λ1.

Diff (Difference Between Event Rates)

This option is displayed only if Enter λ2, Diff, or Ratio for Group 2 = "Diff (Difference Between Event Rates)."

Enter values for the difference between the event rates (Diff = λ2 - λ1). The value of λ2 is equal to λ1 + Diff.

The range of acceptable values is any positive or negative value other than 0 (Diff ≠ 0). You can enter a single value such as *0.5* or a series of values such as *1 to 2 by 0.5*.

RR (Ratio Event Rates)

This option is displayed only if Enter λ2, Diff, or Ratio for Group 2 = "RR (Ratio Event Rates)."

Enter values for the ratio of the event rates (RR = $\lambda 2/\lambda 1$) at which the power is to be calculated. The value of $\lambda 2$ is calculated using $\lambda 1$ x RR.

The range of acceptable values is $RR > 0$ and $RR \neq 1$.

You can enter a single value such as *1.15* or a series of values such as *1.05 to 1.20 by 0.05*.

Example 1 – Calculating Sample Size

Suppose that a study is to be conducted in which $\lambda 1 = 1$, $\lambda 2 = 1.1$ 1.2 1.3 1.4, power = 0.9, and alpha = 0.05. The sample size is to be calculated for a two-sided, large-samples test. The design is to have an equal sample size in each arm.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Tests for the Difference Between Two Poisson Rates** procedure window. You may then make the appropriate entries as listed below, or open **Example 1** by going to the **File** menu and choosing **Open Example Template**.

Option	Value
Design Tab	
Solve For	**Sample Size**
Alternative Hypothesis	**Two-Sided**
Test Statistic	**Large Samples**
Power	**0.90**
Alpha	**0.05**
Group Allocation:	**Equal (N1 = N2)**
λ1 (Event Rate of Group 1)	**1**
Enter λ2, Diff, or Ratio for Group 2	**λ2 (Event Rate of Group 2)**
λ2 (Event Rate of Group 2)	**1.1 1.2 1.3 1.4**

Annotated Output

Click the *Calculate* button to perform the calculations and generate the following output.

Numeric Results

Numeric Results for Testing the Difference Between Two Poisson Rates
Alternative Hypothesis: Two-Sided   Group 1: Control
Test Statistic: Large-Sample   Group 2: Treatment

Power	N1	N2	N	Grp 1 Event Rate λ1	Grp 2 Event Rate λ2	Diff λ2-λ1	Ratio λ2/λ1	Alpha
0.90006	2207	2207	4414	1.00	1.10	0.10	1.1000	0.050
0.90005	578	578	1156	1.00	1.20	0.20	1.2000	0.050
0.90050	269	269	538	1.00	1.30	0.30	1.3000	0.050
0.90070	158	158	316	1.00	1.40	0.40	1.4000	0.050

References
Mathews, Paul. 2010. Sample Size Calculations: Practical Methods for Engineers and Scientists. Mathews Malnar
    and Bailey. Fairport Harbor, OH. www.mmbstatical.com
Smith, P.G. and Morrow, R.H. 1996. Field Trials of Health Intervention in Developing Countries: A Toolbox.
    Macmillan Education. Oxford, England.
Campbell, M.J. and Walters, S.J. 2014. How to Design, Analzse and Report Cluster Randomised Trials in Medicine
   and Health Related Research. John Wiley. New York.

Report Definitions
Power is the probability of rejecting a false null hypothesis. It should be close to one.
N1 and N2 are the number of subjects in groups 1 and 2, respectively.
N is the total sample size. N = N1 + N2.
$\lambda 1$ is the mean event (or incidence) rate of group 1, the control, reference, or baseline group.
$\lambda 2$ is the mean event (or incidence) rate of group 2, the treatment group.
$\lambda 2-\lambda 1$ is the difference between the two event rates.
$\lambda 2/\lambda 1$ is the ratio of the two event rates.
Alpha is the probability of rejecting the null hypothesis when it is true. It should be small.

Summary Statements
A total sample size of 4414 subjects with 2207 in group 1 and 2207 in group 2, each measured
for a fixed duration, achieves 90.006% power to detect a difference of 0.10 between the group 2
(treatment) event rate of 1.10 and the group 1 (control) event rate of 1.00 using a two-sided,
large-samples z-test of the Poisson event-rate difference at a significance level of 0.050.

This report shows the sample size for each of the scenarios.

Plots Section
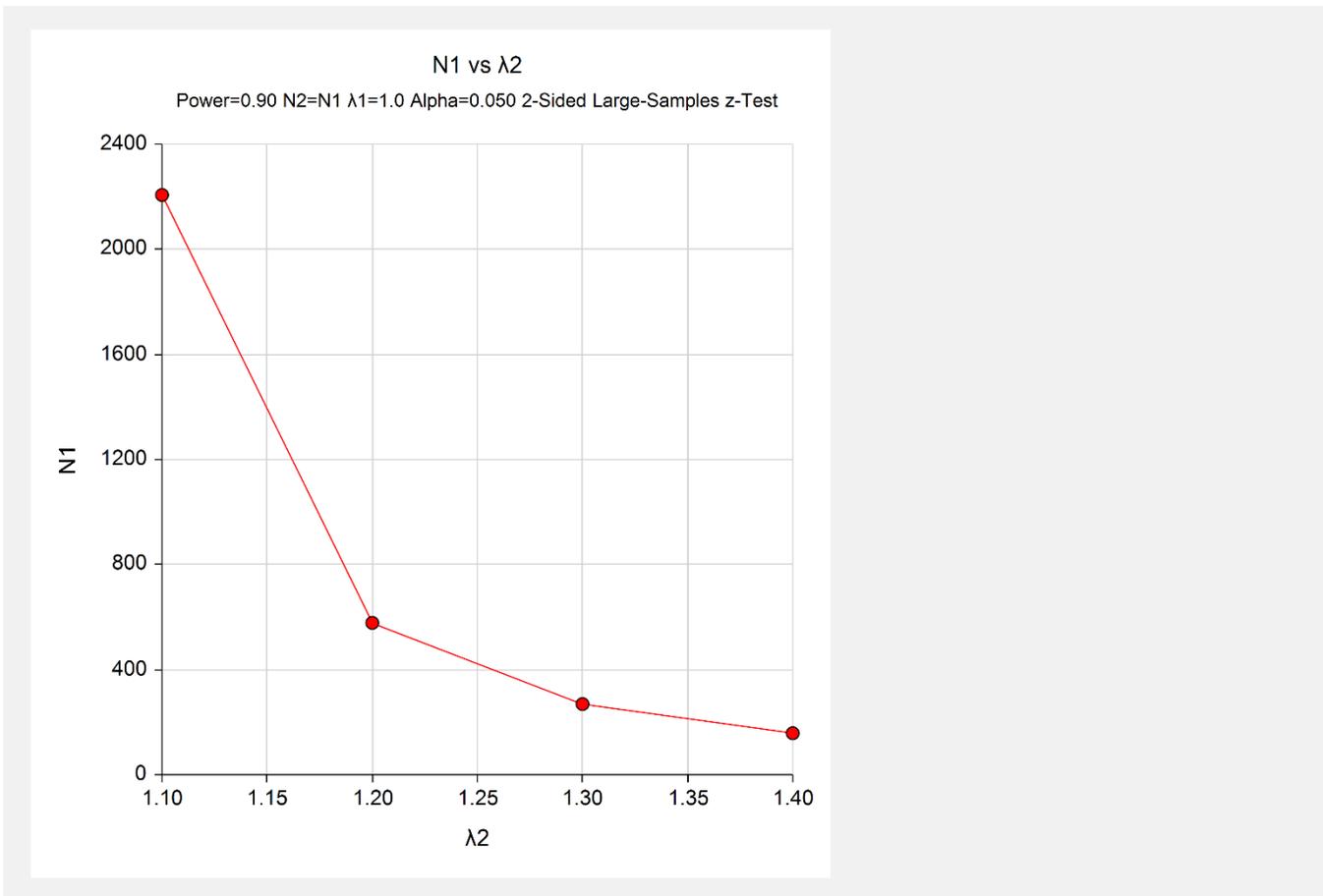
This plot shows the sample size versus the group 2 event rate.

Example 2 – Validation using Mathews (2010)

Mathews (2010) on page 126 presents a power calculation for this test. For the values $\lambda 1 = 10$, $\lambda 2 = 15$, alpha = 0.05, N1 = 8, and N2 = 6, the power is shown to be 0.826 for a one-sided test.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Tests for the Difference Between Two Poisson Rates** procedure window. You may then make the appropriate entries as listed below, or open **Example 2** by going to the **File** menu and choosing **Open Example Template**.

Option   **Value**

Design Tab
Solve For .... **Power**
Alternative Hypothesis .... **One-Sided**
Test Statistic .... **Large Samples**
Alpha **0.05**
Group Allocation: .... **Enter N1 and N2 individually**
N1 .... **8**
N2 .... **6**
λ1 (Event Rate of Group 1) **10**
Enter λ2, Diff, or Ratio for Group 2 .... **λ2 (Event Rate of Group 2)**
λ2 (Event Rate of Group 2) **15**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Numeric Results for Testing the Difference Between Two Poisson Rates
Alternative Hypothesis: One-Sided   Group 1: Control
Test Statistic: Large-Sample   Group 2: Treatment

Power	N1	N2	N	Grp 1 Event Rate λ1	Grp 2 Event Rate λ2	Diff λ2-λ1	Ratio λ2/λ1	Alpha
0.82566	8	6	14	10.00	15.00	5.00	1.5000	0.050

PASS calculates the same power.