Chapter 134

Tests for the Ratio of Two Variances

Introduction

This procedure calculates power and sample size of inequality tests of total (between + within) variabilities from a two-group, parallel design for the case when the ratio assumed by the null hypothesis is equal to one. This routine deals with the case in which the statistical hypotheses are expressed in terms of the ratio of the variances.

Technical Details

This procedure uses the formulation given in Chow, Shao, Wang, and Lokhnygina (2018), pages 217 - 220.

Suppose x_{ij} is the response of the i^{th} group (i = 1, 2) and j^{th} subject (j = 1, …, Ni). The model analyzed in this procedure is

$$x_{ijk} = \mu_i + e_{ij}$$

where μ_i is the treatment effect and e_{ij} is the between-subject error term which is normally distributed with mean 0 and variance $V_i = \sigma_{Bi}^2$. Unbiased estimators of these variances are given by

$$\hat{V}_i = \frac{1}{N_i - 1}\sum_{j=1}^{N_i}\left(x_{ij} - \bar{x}_i\right)^2$$

$$\bar{x}_i = \frac{1}{N_i}\sum_{j=1}^{N_i} x_{ij}$$

A common test statistic to compare variabilities in the two groups is $T = \hat{V}_1/\hat{V}_2$. Under the usual normality assumptions, T is distributed as an F distribution with degrees of freedom $N_1 - 1$ and $N_2 - 1$.

Testing Variance Inequality

The following three sets of statistical hypotheses are used to test for variance inequality

$$H_0: \sigma_1^2/\sigma_2^2 \geq 1 \quad \text{versus} \quad H_1: \sigma_1^2/\sigma_2^2 < 1,$$

$$H_0: \sigma_1^2/\sigma_2^2 \leq 1 \quad \text{versus} \quad H_1: \sigma_1^2/\sigma_2^2 > 1,$$

$$H_0: \sigma_1^2/\sigma_2^2 = 1 \quad \text{versus} \quad H_1: \sigma_1^2/\sigma_2^2 \neq 1,$$

where *one* is the variance ratio assumed by the null hypothesis.

The corresponding test statistics are $T = (\hat{V}_1/\hat{V}_2)$.

Power

The corresponding powers of these three tests are given by

$$\text{Power} = \text{P}\left(F < \left(\frac{1}{R1}\right) F_{\alpha,N_1-1,N_2-1}\right)$$

$$\text{Power} = 1 - \text{P}\left(F < \left(\frac{1}{R1}\right) F_{1-\alpha,N_1-1,N_2-1}\right)$$

$$\text{Power} = \text{P}\left(F < \left(\frac{1}{R1}\right) F_{\alpha/2,N_1-1,N_2-1}\right) + 1 - \text{P}\left(F < \left(\frac{1}{R1}\right) F_{1-\alpha/2,N_1-1,N_2-1}\right)$$

where *F* is the common F distribution with the indicated degrees of freedom, α is the significance level, and *R1* is the value of the variance ratio stated by the alternative hypothesis. Lower quantiles of F are used in the equation.

A simple binary search algorithm can be applied to this power function to obtain an estimate of the necessary sample size.

Example 1 – Finding Sample Size

A company has developed a generic drug for treating rheumatism and wants to compare it to the standard drug in terms of the variance. A parallel-group design is used to test the inequality using a two-sided test.

Company researchers set the variance ratio under the null hypothesis to 1.0, the significance level to 0.05, the power to 0.90, and the actual variance ratio values between 0.5 and 2 (excluding the null hypothesis value, 1.0). They want to investigate the range of required sample size values assuming that the two group sample sizes are equal.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For .... **Sample Size**
Alternative Hypothesis .... **Two-Sided (H1: σ²1/σ²2 ≠ 1)**
Power.. **0.90**
Alpha... **0.05**
Group Allocation .... **Equal (N1 = N2)**
R1 (Actual Variance Ratio) **0.5 0.8 0.9 1.111 1.25 2**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Reports

Numeric Results

Solve For: Sample Size
Hypotheses: H0: $\sigma^2 1/\sigma^2 2 = 1$ vs. H1: $\sigma^2 1/\sigma^2 2 \neq 1$

Power		Sample Size			Actual Variance Ratio	
Target	Actual	N1	N2	N	R1	Alpha
0.9	0.9017	90	90	180	0.500	0.05
0.9	0.9003	847	847	1694	0.800	0.05
0.9	0.9001	3789	3789	7578	0.900	0.05
0.9	0.9000	3796	3796	7592	1.111	0.05
0.9	0.9003	847	847	1694	1.250	0.05
0.9	0.9017	90	90	180	2.000	0.05

Target Power: The desired power value entered in the procedure. Power is the probability of rejecting a false null hypothesis.
Actual Power: The actual power achieved. Because N1 and N2 are discrete, this value is usually slightly larger than the target power.
N1: The number of subjects from group 1.
N2: The number of subjects from group 2.
N: The total number of subjects. N = N1 + N2.
R1: The value of the variance ratio at which the power is calculated. R1 = $\sigma^2 1/\sigma^2 2$.
Alpha: The probability of rejecting a true null hypothesis.

Summary Statements

A parallel two-group design will be used to test whether the variance ratio ($\sigma^2 1 / \sigma^2 2 = \sigma^2 Trt / \sigma^2 Ctrl$) is different from 1 (H0: $\sigma^2 1 / \sigma^2 2 = 1$ versus H1: $\sigma^2 1 / \sigma^2 2 \neq 1$). The comparison will be made using a two-sided, two-sample, variance-ratio F-test, with a Type I error rate (α) of 0.05. To detect a variance ratio of 0.5 with 90% power, the number of subjects needed will be 90 in Group 1 (treatment), and 90 in Group 2 (control).

Dropout-Inflated Sample Size

	Sample Size			Dropout-Inflated Enrollment Sample Size			Expected Number of Dropouts		
Dropout Rate	**N1**	**N2**	**N**	**N1'**	**N2'**	**N'**	**D1**	**D2**	**D**
20%	90	90	180	113	113	226	23	23	46
20%	847	847	1694	1059	1059	2118	212	212	424
20%	3789	3789	7578	4737	4737	9474	948	948	1896
20%	3796	3796	7592	4745	4745	9490	949	949	1898
20%	847	847	1694	1059	1059	2118	212	212	424
20%	90	90	180	113	113	226	23	23	46

Dropout Rate The percentage of subjects (or items) that are expected to be lost at random during the course of the study and for whom no response data will be collected (i.e., will be treated as "missing"). Abbreviated as DR.
N1, N2, and N The evaluable sample sizes at which power is computed. If N1 and N2 subjects are evaluated out of the N1' and N2' subjects that are enrolled in the study, the design will achieve the stated power.
N1', N2', and N' The number of subjects that should be enrolled in the study in order to obtain N1, N2, and N evaluable subjects, based on the assumed dropout rate. After solving for N1 and N2, N1' and N2' are calculated by inflating N1 and N2 using the formulas N1' = N1 / (1 - DR) and N2' = N2 / (1 - DR), with N1' and N2' always rounded up. (See Julious, S.A. (2010) pages 52-53, or Chow, S.C., Shao, J., Wang, H., and Lokhnygina, Y. (2018) pages 32-33.)
D1, D2, and D The expected number of dropouts. D1 = N1' - N1, D2 = N2' - N2, and D = D1 + D2.

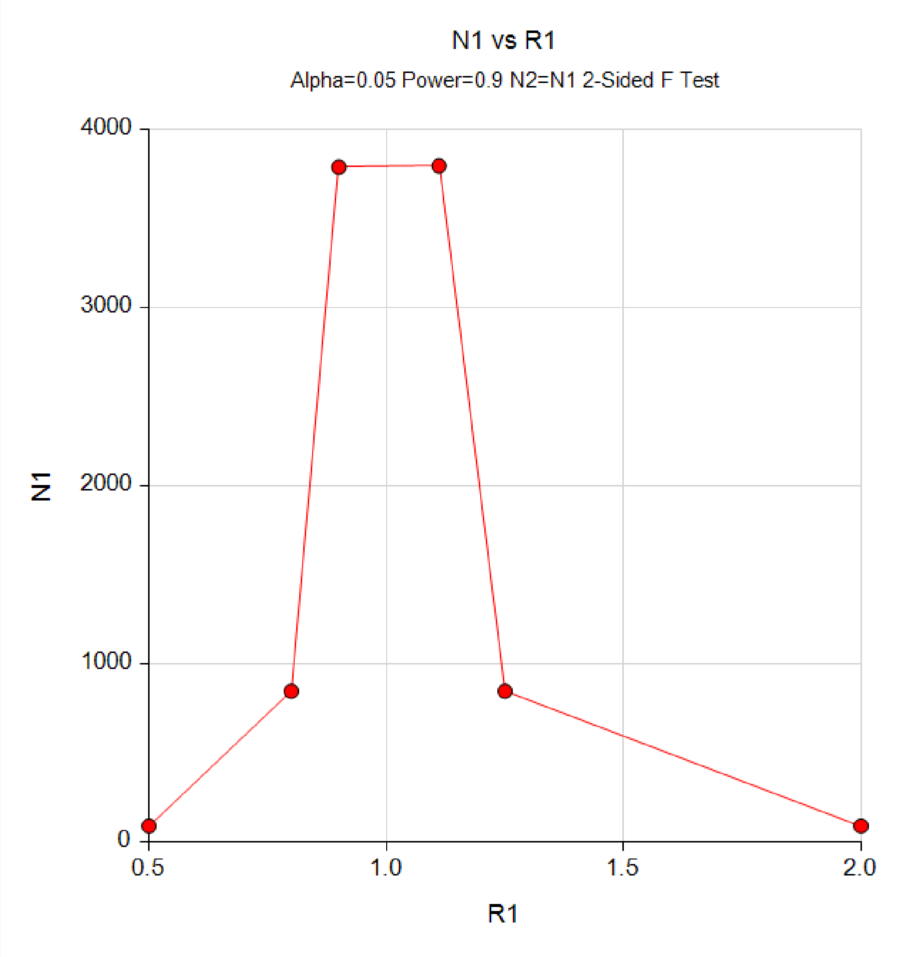
Dropout Summary Statements

Anticipating a 20% dropout rate, 113 subjects should be enrolled in Group 1, and 113 in Group 2, to obtain final group sample sizes of 90 and 90, respectively.

This report gives the sample sizes for the indicated scenarios.

Plots Section

Plots



These plots show the relationship between sample size and R1.

Example 2 – Validation using Davies (1971)

Davies (1971) page 41 presents an example with *R1*= 4, *Alpha* = 0.05, and *Power* = 0.99 in which the sample sizes, *N1* and *N2*, are calculated to be 36 assuming a one-sided hypothesis. We will run this example through **PASS**.

Setup

If the procedure window is not already open, use the PASS Home window to open it. The parameters for this example are listed below and are stored in the **Example 2** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Design Tab

Solve For ....**Sample Size**
Alternative Hypothesis ....**One-Sided (H1: σ²1/σ²2 > 1)**
Power...**0.99**
Alpha..**0.05**
Group Allocation ....**Equal (N1 = N2)**
R1 (Actual Variance Ratio) **4**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Solve For: Sample Size
Hypotheses: H0: σ²1/σ²2 ≤ 1 vs. H1: σ²1/σ²2 > 1

Power		Sample Size			Actual Variance Ratio	
Target	Actual	N1	N2	N	R1	Alpha
0.99	0.9914	36	36	72	4	0.05

The sample sizes match the results in Davies (1971).